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A Special Class of Distributions on n-Balls and its Applications to Stochastic Simulation

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Abstract

In this work we propose a family of multivariate distributions on n-balls, which is closed with respect to forming marginal and conditional distributions. We analyze its basic properties and construct a class of algorithms generating random vectors on the surface and in the interior of n-balls. The numerical experiments showed that the proposed method for generating uniformly distributed points in the interior of n-balls is, to our best knowledge, more efficient than other published methods.

Keywords: Uniform distribution, Beta distribution, Normal distribution, n-Sphere, n-Ball, Simulation.

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Introduction

With the advent of computers into the science, new ways of solving practical problems emerged. In mathematics, branches based on numerical computing started to develop with full speed, among other things the simulation methods. We are often not able to find solutions of real optimization problems (optimization of traffic lights, the queueing theory, optimization of "ugly" functions on bounded areas) in an analytic form. For example in the queueing theory we have distributions for costumers arrivals, time of servicing, patience of a customer, decision of changing a queue or leaving the system. An approximately optimal solution can be found using simulations.

The prerequisite of the use of the simulation methods is to find effective ways of generating elements from a particular set such that the elements have required distribution.

In this work we focus our attention on multivariate distributions on balls, especially on the uniform distribution on the surface and in the interior of the n-ball. Motivation for this problem is, for example, optimization of functions on bounded sets. Since we generate uniformly on the feasible set, the objective function needs not necessarily be differentiable or continuous etc. This tasks can be divided into three groups with increasing difficulty level:

- Optimization of functions on the surface and in the interior of n-balls:

  $$\min\{f(\mathbf{x}) | \|\mathbf{x}\| = r,\ r > 0\},$$
  $$\min\{f(\mathbf{x}) | \|\mathbf{x}\| \leq r,\ r > 0\}.$$

  In random search optimization methods we generate random points and compare output values.

- Optimization on convex bounded areas (polytops as a special case): Let $\mathcal{X}$ be an n-dimensional convex bounded area and let $\mathbf{x}^0$ be an initial interior point. Generating from uniform distribution on the surface of the unit n-ball we get a random direction $\mathbf{s}^0$. Let $\lambda^0 > 0$ be such that $\overline{\mathbf{x}}^0 = \mathbf{x}^0 + \lambda_0 \mathbf{s}^0 \in \mathcal{X}$. Let U^0 be uniformly distributed on the unit interval $(0, 1)$. Setting $\mathbf{x}^1 = U^0 \mathbf{x}^0 + (1 - U)\overline{\mathbf{x}}^0$ (random point on a line segment) we get another interior point and we repeat the procedure. Using this method we obtain a Markovian process which is asymptoticaly uniformly distributed on $\mathcal{X}$. The key factor here is the ability to generate a random direction. Generating uniformly distributed points we compare output values of the objective function.

- Optimization on general bounded areas: The idea is the same as in the previous case. The problem is if the area is concave and/or contains "holes". Then it might happen that we generate on more than one line segment.

The most important fact about the mentioned algorithms is that they are polynomial, in contrast to the classic rejection method, which is exponential. These methods are described in [4, sec.5.13, p.384].

The main result of this work is to propose a new family of multivariate distributions which is closed with respect to forming marginal and conditional distribution. As we show, the family induces a class of algorithms, which also contains the method analyzed in [17]. The class includes also a method for generating points uniformly distributed in the interior of the unit n-ball that is, to our best knowledge, more efficient than other published methods.

This work is organized as follows: Chapter 1 describes some univariate and multivariate distributions. We highlight the uniform distribution on the surface of the unit n-ball (we often call unit n-sphere) and in the interior of the unit n-ball and the Beta distribution with its properties. Chapter 2 is dedicated to basic methods of generating special distribution families. Chapter 3 defines a special family of multivariate distributions and its applications to stochastic simulations. In Chapter 4 we compare algorithms that result from this work with several algorithms known from the literature. Chapter 5 contains some cited propositions, source codes and result tables.

Notation

$\widetilde{\boldsymbol{\xi}}$	a general orthogonal projection of $\boldsymbol{\xi}$
$\widetilde{\boldsymbol{\xi}}_{\mathcal{V}}$	an orthogonal projection of $\boldsymbol{\xi}$ to a linear subspace $\mathcal{V}$;
$\mathcal{S}_n$, $\mathcal{S}_n(\rho)$	the unit n-sphere and n-sphere with radius ρ, respectively ($\mathcal{S}_{\mathcal{N}} = \mathcal{S}_n,\ n = \dim(\mathcal{N})$);
$\mathcal{B}_n$, $\mathcal{B}_n(\rho)$	the unit n-ball and n-ball with radius ρ;
$\xi \overset{d}{\sim} \eta$	equivalency of ξ and η with respect to their distribution.
$\|.\|$	Euclidian vector norm, i.e. $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2}$

All linear spaces are assumed to be vector linear spaces.

Chapter 1

Special Families of Random Variables and Vectors

1.1 Uniform Distribution on the Unit n-Sphere and in the Unit n-Ball

To describe a spherical distribution analytically is rather complicated. For example, the unit n-sphere has dimension $n-1$, so the measure of such set is zero. Therefore, this distributions are defined on sets describing their "character". Let $\mathcal{S}_n$ be the unit n-sphere, i.e

$$\mathcal{S}_n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}.$$

Then the density function of the uniform distribution on the unit n-sphere relative to the Lebesgue measure in $\mathcal{S}_n$ is

$$f(\mathbf{x}) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}},\ \mathbf{x} \in \mathcal{S}_n,$$

what represents the reciprocal value of the surface of the unit n-sphere. It means, that the probability that $\boldsymbol{\xi}$ is to occur in the set $\Omega \subseteq \mathcal{S}_n$ is the ratio of the surface of Ω and the surface of the sphere (see fig. 1.1).

The unit n-ball

$$\mathcal{B}_n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\},$$

compared with the n-sphere, has nonzero Lebesgue measure on $\mathbb{R}^n$. Even though its distribution is not easy to described analytically, its density is defined in $\mathbb{R}^n$

$$f(\mathbf{x}) = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}},\ \mathbf{x} \in \mathcal{B}_n.$$

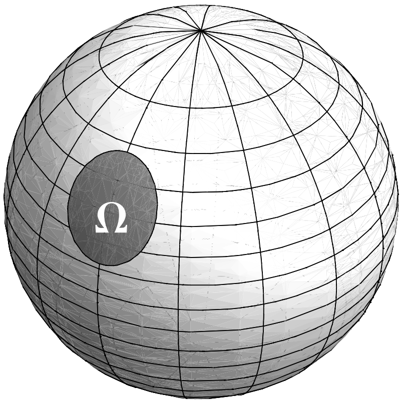

Figure 1.1: Uniform distribution on the unit sphere. The probability that the random vector is to occur in $\Omega \subseteq \mathcal{S}_n$ is equal to ratio between the surface of the area and the sphere.

1.2 Exponential Distribution

We say that a random variable ξ has the exponential distribution with parameter λ (denote $\xi \sim Exp(\lambda)$), if the density of the uniform distribution on $\mathcal{B}_n$ is

$$f(x;\lambda) = \frac{1}{\lambda}e^{-\frac{x}{\lambda}},\ x \geq 0,\ \lambda > 0.$$

1.3 Normal Distribution

We say that a random variable ξ has the normal distribution with mean μ a variance σ^2 (denote $\xi \sim N(\mu, \sigma^2)$), if its density function is

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \sigma > 0.$$

If $\mu = 0,\ \sigma = 1$ we say the random variable has the standard normal distribution.

1.4 χ^2 Distribution

Let $\xi_1, ..., \xi_n \sim N(0,1)$ be independent. Let

$$\eta = \sum_{i=1}^{n} \xi_i^2.$$

Then we say that η is Chi-square distributed with n degrees of freedom (notation $\eta \sim \chi_n^2$). The density function is

$$f(x;n) = \frac{1}{\Gamma(\frac{n}{2})2^{n/2}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}},\ x > 0,\ n \in \mathbb{N}.$$

1.5 Γ Distribution

We say that ξ has the Gamma distribution with parameters k and θ (notation $\xi \sim \Gamma_{k,\theta}$), if its density function is

$$f(x;k,\theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}},\ x > 0,\ \theta > 0,\ k > 0$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t,$$

is the so-called Gamma function. Note that the χ_n^2 distribution is only a special case of the Γ distribution for $k = n/2$ and $\theta = 2$.

1.6 Fisher-Snedecor's F Distribution

Let $\eta_1 \sim \chi_n^2$, $\eta_2 \sim \chi_m^2$ and

$$\zeta = \frac{\eta_1/n}{\eta_2/m}.$$

Then we say, that ζ has F distribution with n and m degrees of freedom (notation $\zeta \sim F_{n,m}$). The density function of F distribution is

$$f(x;n,m) = \frac{\left(\frac{nx}{nx+m}\right)^{\frac{n}{2}} \left(1 - \frac{nx}{nx+m}\right)^{\frac{m}{2}}}{xB(\frac{n}{2}, \frac{m}{2})},$$

where B is Beta function (see the next section).

1.7 Beta Distribution

The Beta distribution has been known since times of Sir Isaac Newton.

We say that ξ has the Beta distribution with parameters α and β (notation $\xi \sim B_{\alpha,\beta}$), if its density function is

$$f(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0 \le x \le 1,\ \alpha > 0,\ \beta > 0,$$

where

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1},$$

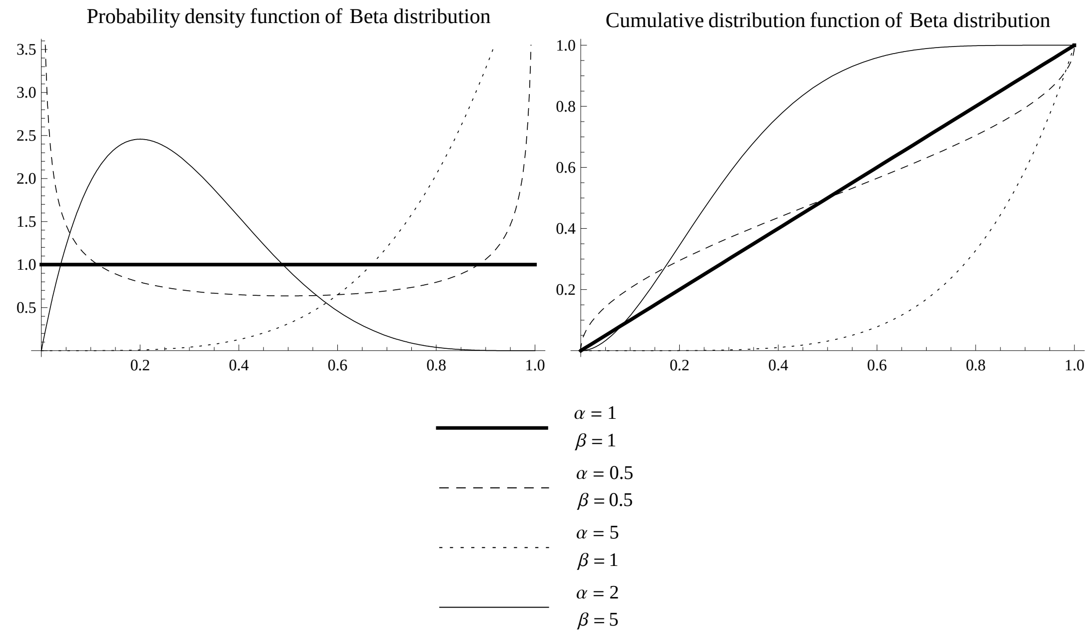

Figure 1.2: The probability density function and the cumulative distribution function of Beta distribution for different parameters.

is the Beta function. In this work we also need to deal with a limit case for $\beta = 0$, therefore we define this limit case of the Beta distribution as follows:

$$P(\xi = 1; \beta = 0) = 1 \; \forall \alpha > 0.$$

Thus, for $\beta = 0$ we have a discrete distribution, which gives 1 with probability 1.

The advantages of the Beta distribution is variability and bounded support (the unit interval can be scaled to any interval). Remark that

$$f(x; \alpha, \beta) = f(1 - x; \beta, \alpha),$$

so, if $\xi \sim B_{\alpha,\beta}$, then $1 - \xi \sim B_{\beta,\alpha}$.

Remark: For $\alpha = \beta = 1$ we have the uniform distribution on $(0, 1)$.

The Mean

$$\mathrm{E}[\xi] = \frac{\alpha}{\alpha + \beta},$$

the variance

$$\mathrm{Var}[\xi] = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)},$$

and the characteristic function of the Beta distribution

$$\mathrm{E}[e^{it\xi}] = {}_1F_1(\alpha; \alpha + \beta; it),$$

where

$$ {}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}, $$

is the a confluent hypergeometric function of the first kind and $(a)_n = a(a+1)(a+2)...(a+n-1) = \Gamma(a+n)/\Gamma(n)$. The kth central moment is given by

$$ \mathrm{E}[(\xi - E[\xi])^k] = \left(-\frac{\alpha}{\alpha+\beta}\right)^k {}_2F_1\left(-k, \alpha; \alpha+\beta; \frac{\alpha+\beta}{\alpha}\right), $$

where

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}, $$

is the Gauss hypergeometric function.

In Bayesian statistics we can observe an interesting property of the Beta distribution. Suppose ξ has the binomial distribution with parameters n a p, i.e.

$$ P(\xi = k) = \binom{n}{k} p^k (1-p)^{n-k}, $$

and the prior distribution of p is the Beta distribution with parameters α, $\beta > 0$ (denote the density function $\pi(\theta)$). The Bayes formula gives

$$ f_\zeta(x|\eta = y) = \frac{f_\eta(y|\zeta = x) f_\zeta(x)}{\int_{-\infty}^{\infty} f_\eta(y|\zeta = \nu) f_\zeta(\nu) \mathrm{d}\nu}, $$

where $f_\zeta(x)$ is the prior and $f_\zeta(x|\eta = y)$ is the posterior density function. Setting the binomial probability mass function and the density function of the Beta distribution into Bayes formula we get

$$ \pi(\theta|\xi = k) = \frac{\binom{n}{k}\theta^k(1-\theta)^{n-k}\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\int_{-\infty}^{\infty}\binom{n}{k}\nu^k(1-\nu)^{n-k}\frac{1}{B(\alpha,\beta)}\nu^{\alpha-1}(1-\nu)^{\beta-1}\mathrm{d}\nu} = \frac{\theta^{k+\alpha-1}(1-\theta)^{n-k+\beta-1}}{B(k+\alpha, n-k+\beta)}, $$

i.e., the posterior distribution of parameter p is Beta.

There are many variations a and generalizations of the Beta distribution, e.g see [12] or [5]. A way how to generalize the Beta distribution is the Gauss hypergeometric distribution [2], defined by

$$ f(x;\alpha,\beta,\gamma,\lambda) = \frac{1}{B(\alpha,\beta)_2F_1(\gamma,\alpha;\alpha+\beta;\kappa)} x^{\alpha-1}(1-x)^{\beta-1}(1-\kappa x)^{-\gamma},\ 0 \le x \le 1, $$

If $\gamma = 0$ or $\kappa = 0$, we have the density function of the Beta distribution with parameters α, β.

Below we formulate a several of interesting propositions about the properties of Beta distribution.

Lemma 1.7.1. *Let $\xi_k \sim \chi_k^2$. Then*

$$ \frac{\xi_m}{\xi_m + \xi_n} \sim B_{\frac{m}{2},\frac{n}{2}}. $$

Proof. Denote $\gamma_{k,\theta} \sim \Gamma_{k,\theta}$. According to [4, sec.3.15, p.200] it holds that

$$\frac{\gamma_{\alpha,1}}{\gamma_{\alpha,1} + \gamma_{\beta,1}} = \frac{1}{1 + \frac{\gamma_{\beta,1}}{\gamma_{\alpha,1}}} \sim B_{\alpha,\beta}.$$

We want to prove

$$\frac{\gamma_{\frac{q}{2},2}}{\gamma_{\frac{q}{2},2} + \gamma_{\frac{p}{2},2}} = \frac{1}{1 + \frac{\gamma_{\frac{p}{2},2}}{\gamma_{\frac{q}{2},2}}} \sim B_{\frac{q}{2},\frac{p}{2}}.$$

The relationship is true if and only if the distribution of

$$\nu = \frac{\gamma_{\alpha,\theta}}{\gamma_{\beta,\theta}}$$

does not depend on parameter θ, i.e. the density function f_ν of ν does not depend on θ. Using the formula for fraction of two random variables we get

$$\begin{aligned} f_\nu(x;\alpha,\beta,\theta) &= \int_0^\infty y \frac{1}{\Gamma(\alpha)\theta^\alpha}(xy)^{\alpha-1} e^{-\frac{xy}{\theta}} \frac{1}{\Gamma(\beta)\theta^\beta} y^{\beta-1} e^{-\frac{y}{\theta}} \mathrm{d}y \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} \int_0^\infty y^{(\alpha+\beta)-1} e^{-\frac{(1+x)y}{\theta}} \mathrm{d}y \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} \Gamma(\alpha+\beta) \left(\frac{\theta}{1+x}\right)^{\alpha+\beta} \\ &= \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1+x)^{-(\alpha+\beta)}. \end{aligned}$$

□

Theorem 1.7.2. *Let s, p, $q \in \mathbb{R}$, $0 < s < q < p$ a $\beta_1 \sim B_{q,p-q}$, $\beta_2 \sim B_{s,q-s}$ be independent and $\nu = \beta_1.\beta_2$. Then*

$$\nu \sim B_{s,p-s}.$$

Proof. The density function of product of two random variables ([14, p.188, (2)]) gives

$$f_\nu(x;p,q,s) = K \int_x^1 \left(\frac{x}{y}\right)^{q-1} \left(1 - \frac{x}{y}\right)^{p-q-1} y^{s-2}(1-y)^{q-s-1} \mathrm{d}y, \ \ 0 \le x \le 1,$$

where $\frac{1}{B(q,p-q)B(s,q-s)}$. Multiplying f_ν by

$$\frac{x^{s-1}(1-x)^{p-s-1}}{x^{s-1}(1-x)^{p-s-1}},$$

we have

$$f_\nu(x;p,q,s) = \\ K x^{s-1}(1-x)^{p-s-1} \underbrace{\int_x^1 \frac{y^{s-2}(1-y)^{q-s-1}}{x^{s-1}(1-x)^{p-s-1}} \left(\frac{x}{y}\right)^{q-1} \left(1 - \frac{x}{y}\right)^{p-q-1} \mathrm{d}y}_{\Psi(x)}.$$

We compute $\Psi(x)$. First we have to eliminate x in integration boundaries.

$$\begin{aligned}\Psi(x) &= \int_x^1 \frac{y^{s-2}(1-y)^{q-s-1}}{x^{s-1}(1-x)^{p-s-1}}\left(\frac{x}{y}\right)^{q-1}\left(1-\frac{x}{y}\right)^{p-q-1}\mathrm{d}y \\ &\qquad \begin{array}{rcl} z &=& \frac{1-y}{1-x} \\ \text{substitution} \quad y &=& 1-z(1-x) \\ \mathrm{d}y &=& -(1-x)\mathrm{d}z \end{array} \\ &= \int_0^1 x^{q-s}z^{q-s-1}[1-z(1-x)]^{s-p}(1-z)^{p-q-1}\mathrm{d}z = \\ &= \int_0^1 x^{q-s}z^{(q-s)-1}(1-z)^{(p-s)-(q-s)-1}[1-(1-x)z]^{s-p}\mathrm{d}z.\end{aligned}$$

Using [1, 15.3.1, p.558] (see Appendix)

$$\Psi(x) = B(p-q, q-s)x^{q-s}{}_2F_1(p-s, q-s, p-s, 1-x),$$

and [1, 15.1.8, p.556] (see Appendix)

$$\Psi(x) = B(p-q, q-s)x^{q-s}[1-(1-x)]^{-(q-s)}.$$

Setting to f_ν

$$f_\nu(x; p, q, s) = \frac{1}{B(s, p-s)}x^{s-1}(1-x)^{p-s-1}.$$

□

In [8, thm.1, p.402] is mentioned an equivalent theorem (without the proof, we had to prove it ourselves) in the following form

$$\beta_1 \sim B_{a,b},\ \beta_2 \sim B_{a+b,c} \Rightarrow \beta_1.\beta_2 \sim B_{a,b+c},$$

which we get if we set $q := a+b,\ p-q := c,\ s := a,\ q-s := b$ in Theorem 1.7.2.

Lemma 1.7.3. *Let ξ be a random variable, then*

$$\xi \sim U^p_{(0,1)} \Leftrightarrow \xi \sim B_{\frac{1}{p},1}$$

Proof. Let $\xi = \eta^p$, $\eta \sim U_{(0,1)}$. Then

$$\begin{aligned} F_\xi &= \Pr(\xi < x) = \Pr(\eta^p < x) = \Pr(\eta < x^{\frac{1}{p}}) = F_\eta(x^{\frac{1}{p}}), \\ f_\xi &= f_\eta(x^{\frac{1}{p}})x^{\frac{1}{p}-1}\frac{1}{p} = x^{\frac{1}{p}-1}\frac{1}{p},\ 0 \le x \le 1. \end{aligned}$$

On the other hand

$$f_{B_{\frac{1}{p},1}}(x) = \frac{1}{B(\frac{1}{p},1)}x^{\frac{1}{p}-1},\ 0 \le x \le 1.$$

□

Denote $\beta_{\alpha,\beta}$ independent $B_{\alpha,\,\beta}$ distributed random variables. We say that random variables ξ, η are distributionally equivalent (denote $\xi \stackrel{d}{\sim} \eta$), if they have the same density.

Now we compute $\beta_{a,1}.\beta_{b,1}$, without loss of generality $a > b$. Denote $q = a,\ s = b,\ p - q = 1,\ q - s = 1$. Then

$$\begin{aligned} p - s &= 2, \\ p &= 1 + q = 1 + a, \\ s &= b = p - 2 = 1 + a - 2 = a - 1 > 0. \end{aligned}$$

According to Lemma 1.7.3 we get $\beta_{a,1}.\beta_{a-1,1} \stackrel{d}{\sim} \beta_{a-1,2}$. Let $a := a+1$, then $\beta_{a+1,1}.\beta_{a,1} \stackrel{d}{\sim} \beta_{a,2}$. It leads us to compute

$$\beta_{a+n,1}.\beta_{a,n} \stackrel{d}{\sim} \beta_{a+n,(a+n+1)-(a+n)}.\beta_{a,(a+n)-a}.$$

Using Lemma 1.7.3 a we have the following equation

$$\beta_{a+n,1}.\beta_{a,n} \stackrel{d}{\sim} \beta_{a,n+1}. \tag{1.1}$$

Using equation (1.1) we decompose the following product

$$\begin{aligned} \beta_{a,n+1} &\stackrel{d}{\sim} \beta_{a+n,1}.\beta_{a,n} \stackrel{d}{\sim} \beta_{a+n,1}(\beta_{a+n-1,1}.\beta_{a,n-1}) \\ &\stackrel{d}{\sim} \beta_{a+n,1}.\beta_{a+n-1,1}(\beta_{a+n-2,1}.\beta_{a,n-2}) = \dots . \end{aligned}$$

Induction gives the decomposition until $\beta_{a,1}$. The result is summarized in the following theorem.

Theorem 1.7.4 ([7]). *Let $a > 0$ and*

$$\beta_i \sim B_{a+i,1},$$

be independent. Then

$$\nu = \prod_{i=0}^{n-1} \beta_i \sim B_{a,n},\ i = 0, 1, ..., n-1 \tag{1.2}$$

Using Lemma 1.7.3 we write (1.2) in the following form

$$\nu = \prod_{i=0}^{n-1} \eta_i^{\frac{1}{a+i}} \sim B_{a,n},\ \text{if}\ \eta_i \stackrel{iid}{\sim} U_{(0,1)}. \tag{1.3}$$

Chapter 2

Selected Topics from the Simulation Methods

The basis of the simulation methods theory is ability to generate random numbers, i.e. numbers uniformly distributed on $(0, 1)$. So, in the following we assume we are able to generate from $U_{(0,1)}$.

2.1 Inverse Transform Theorem

Since we are able to generate from $U_{(0,1)}$, the following theorem enables us to generate from any univariate distribution.

Theorem 2.1.1. [9, chapt.2, s.21] *Let ξ be a random variable with the cumulative distribution function $F(x)$ and set $G(y) = \sup\{x : F(x) \leq y\}$ for $y \in (0, 1)$. If $U \sim U_{(0,1)}$, then the random variable $G(U)$ has cumulative distribution function $F(x)$.*

Proof. We show that $F(x) \leq y \Leftrightarrow x \leq G(y)$. Let $F(x) \leq y$, then directly from the definition of $G(y)$ results that $x \leq G(y)$. Let $x \leq G(y)$ and $\varepsilon > 0$. According to definition of supreme exists z_ε that $x - \varepsilon \leq z_\varepsilon$ and $F(z_\varepsilon) \leq y$. Since F is non-decreasing, it holds that $F(x-\varepsilon) \leq F(z_\varepsilon) \leq y$. Since $\varepsilon > 0$ and distribution function is left-continuous, it holds that $F(x) \leq y$.

Now we compute

$$P(G(U) < x) = 1 - P(G(U) \geq x) = 1 - P(U \geq F(x)) = P(U < F(x)) = F(x).$$

□

Remark: If F is invertible, then $G = F^{-1}$. Then we have $\xi = F^{-1}(U)$ has distribution function F.

Theorem 2.1.2 (Exponential distribution generator). *Let*

$$\xi = -\lambda \ln(U),\ U \sim U_{(0,1)}.$$

Then $\xi \sim Exp(\lambda)$

Proof. Since the cumulative distribution function of exponential distribution is invertible, we have

$$\begin{aligned} y &= 1 - e^{-\frac{x}{\lambda}}, \\ 1 - y &= e^{-\frac{x}{\lambda}}, \\ \ln(1-y) &= -\frac{x}{\lambda}, \\ x &= -\lambda \ln(1-y). \end{aligned}$$

Using Inverse Transform Theorem we get

$$\xi = -\lambda \ln(1 - U^*) \sim Exp(\lambda),\ U^* \sim U_{(0,1)}.$$

Obviously, if $U^* \sim U_{(0,1)}$, then $U = 1 - U^* \sim U_{(0,1)}$, so

$$\xi = -\lambda \ln(U),\ U \sim U_{(0,1)}.$$

□

2.2 Simulation of Special Families of Random Variables

The inverse transform theorem gives us a general method of how to generate random variables with distribution F. However, not all distribution functions are invertible and/or easy-to-invert. As an example we have the normal distribution, where the distribution function is rather complicated, since we integrate the Gaussian function, and computing the value of the inverse function at a specified point is an even more complicated problem. Unfortunately, this is not an exception. Many special distributions do not have a "nice" cumulative distribution function, we mention Beta, Γ, χ^2, Student's t, or Fisher-Snedecor's F distribution. One usually needs to explore another ways of how to simulate random variables from these distributions. In this section we mention those methods, which we use later.

2.2.1 Normal and Multidimensional Normal Distribution

[15, kap. 5, s.78-82] Assume that ξ a η are independent $N(0,1)$ distributed random variables and let r and ϕ be polar coordinates of the vector $(\xi, \eta)'$. Then $R^2 = \xi^2 + \eta^2$ and $\tan(\Phi) = \eta/\xi$. Since the components of $(\xi, \eta)'$ are independent, the joint density function is

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}.$$

To determine the joint density function of $(R^2, \Phi)'$, denote $f(r, \phi)$, we transform the variables

$$\begin{aligned} r &= x^2 + y^2, \\ \phi &= \arctan(y/x). \end{aligned}$$

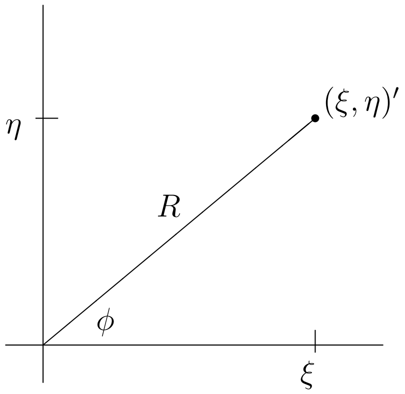

Figure 2.1: Polar coordinates of a random vector in $\mathbb{R}^2$.

The Jacobian matrix of this transformation is

$$\begin{pmatrix} 2x & 2y \\ -\dfrac{1}{1+\frac{y^2}{x^2}}\cdot\dfrac{y}{x^2} & \dfrac{1}{1+\frac{y^2}{x^2}}\cdot\dfrac{1}{x} \end{pmatrix},$$

and Jacobian determinant is 2. So Jacobian of the inverse transformation is 1/2, thus, the joint density of $(R^2, \Phi)'$ is

$$f(r,\phi) = \frac{1}{2}\frac{1}{2\pi}e^{-\frac{r}{2}},\ 0 < d < \infty,\ 0 < \phi < 2\pi.$$

We observe that $f(r,\phi)$ is product of densities of uniform distribution on interval $(0, 2\pi)$, namely $(2\pi)^{-1}$, and exponential distribution with parameter parameter $\lambda = 2$, namely $\frac{1}{2}e^{-r/2}$. So, R^2 and Φ are independent, $R^2 \sim Exp(2)$ and $\Phi \sim U_{(0,2\pi)}$. For inverse transformation we get

$$\begin{aligned} \xi &= R\cos(\Phi), && (2.1)\\ \eta &= R\sin(\Phi). && (2.2) \end{aligned}$$

Theorem 2.2.1 (Box-Muller generator)**.** *Let $U_1, U_2 \sim U_{(0,1)}$ be independent. Then for ξ, η defined by*

$$\begin{aligned} \xi &= \sqrt{-2\ln(U_1)}\cos(2\pi U_2),\\ \eta &= \sqrt{-2\ln(U_1)}\sin(2\pi U_2), \end{aligned} \qquad (2.3)$$

it holds that ξ and η are independent and $N(0,1)$ distributed random variables.

Proof. Let $U_1, U_2 \sim U_{(0,1)}$ Since $R^2 \sim Exp(2)$, then according to Theorem 2.1.2 at the page 11 it holds that $R^2 = -2\ln(U_1)$, thus $R = \sqrt{-2\ln(U_1)}$. We have $2\pi U_2 \sim U_{(0,2\pi)}$. Setting into (2.1) and (2.2) we proved the theorem. □

Remark: Transformations (2.3) are called Box-Muller transformations.

Since Box-Muller transformations require evaluating trigonometric functions, the method in Theorem 2.2.1 is not very efficient. Denote $A = 2U_1 - 1$, $B = 2U_2 - 1$,

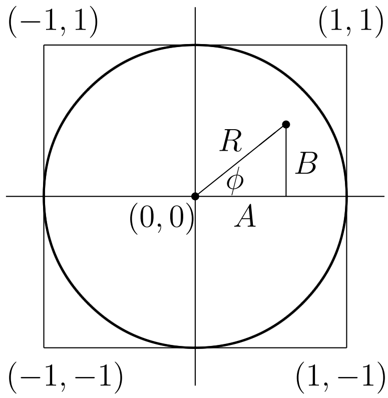

Figure 2.2: Uniform distribution in the square. If the realization of a random vector occurs in the unit circle, we accept it, otherwise we reject it. (the so-called rejection method).

where $U_1, U_2 \sim U_{(0,1)}$ are independent. Then $A, B \sim U_{(-1,1)}$ are independent. If the realization of $(A, B)'$ occurs in the unit circle, i.e. $A^2 + B^2 \leq 1$, then we accept the pair, otherwise we generate a new pair. Using this rejection method we get realizations, which are uniformly distributed in the unit circle. Denote R, Φ the polar coordinates of this pair (see figure 2.2), it is obvious that R, Φ are independent, furthermore $R^2 \sim U_{(0,1)}$ and $\Phi \sim U_{(0,2\pi)}$. We can compute sine and cosine of the angle Φ as following

$$\begin{aligned} \sin(\Phi) &= \tfrac{B}{\sqrt{A^2+B^2}}, \\ \cos(\Phi) &= \tfrac{A}{\sqrt{A^2+B^2}}. \end{aligned} \tag{2.4}$$

Let $U \sim U_{(0,1)}$. Setting (2.4) into Box-Muller generator we get the following equations

$$\begin{aligned} \xi &= \sqrt{-2\ln(U)}\frac{A}{\sqrt{A^2+B^2}}, \\ \eta &= \sqrt{-2\ln(U)}\frac{B}{\sqrt{A^2+B^2}}. \end{aligned}$$

Unfortunately we have to generate 3 random variables uniformly distributed on $(0, 1)$. Since the angle Φ and radius R are independent and $R^2 \sim U_{(0,1)}$, we can use R^2 instead of U and we have

$$\begin{aligned} \xi &= \sqrt{\frac{-2\ln(R^2)}{R^2}}A, \\ \eta &= \sqrt{\frac{-2\ln(R^2)}{R^2}}B. \end{aligned}$$

Algorithm 2.2.2 (Polar method)**.** *The algorithm below generates a pair ξ, η of independent $N(0, 1)$ distributed random variables.*

function: $[\xi, \eta] = randN01()$
(P1) *Generate* $U_1, U_2 \sim U_{(0,1)}$

$A := 2U_1 - 1, B := 2U_2 - 1$
$R^2 := A^2 + B^2$
if $R^2 > 1$
    goto *P1*
end

$$\xi = \sqrt{\frac{-2\ln(R^2)}{R^2}}A$$

$$\eta = \sqrt{\frac{-2\ln(R^2)}{R^2}}B$$

Remark: Let $\xi \sim N(0,1)$. If we want to generate a realization $\nu \sim N(\mu, \sigma^2)$, then we set $\nu = \sigma\xi + \mu$. Then $\mathrm{E}[\nu] = \mathrm{E}[\sigma\xi + \mu] = \sigma\mathrm{E}[\xi] + \mu = \mu$ and $\mathrm{Var}[\nu] = \mathrm{Var}[\sigma\xi + \mu] = \sigma^2\mathrm{Var}[\xi] = \sigma^2$.

Remark: The Box-Muller generator gives $\xi^2 + \eta^2 = -2\ln(U)$, $U \sim U_{(0,1)}$. Since ξ and η are independent, then $\nu = \xi^2 + \eta^2 \sim \chi^2_2$. So $-2\ln(U)$ generates from χ^2_2.

[4, sec.3.24, p.223] Suppose we want to generate from $N_n(\boldsymbol{\mu}, \mathbf{\Sigma})$. Let $\boldsymbol{\xi} \sim N_n(\mathbf{0}, \mathbf{I})$ and $\mathbf{\Sigma} = \mathbf{C}\mathbf{C}'$. Then $\mathbf{C}\boldsymbol{\xi} + \boldsymbol{\mu} \sim N_n(\mathbf{0}, \mathbf{I})$. Now we show a technique how to generate from $N_n(\boldsymbol{\mu}, \mathbf{\Sigma})$ efficiently.

Algorithm 2.2.3. *This algorithm computes such lower triangular matrix* $\mathbf{C}$ *that* $\mathbf{\Sigma} = \mathbf{C}\mathbf{C}'$*. We denote* $\mathbf{\Sigma} = (\sigma_{ij})$, $\mathbf{C} = (c_{ij}) \in \mathbb{R}^{n\times n}$.

function: $[\mathbf{C}] = Cholesky(\mathbf{\Sigma})$
  $a := \sqrt{\sigma_{11}}$
  $i := 1$
  until $i > n$
    $c_{i1} := \sigma_{i1}/a$
    $i := i + 1$
  end $c_{22} := \sqrt{\sigma_{22} - c_{21}^2}$
  $i := 3$
  while $i \leq n$
    $j := 2$
    while $j < i$
      $c_{ij} := (\sigma_{ij} - \sum_{m=1}^{j-1} c_{im}c_{jm})/c_{jj}$
      $j := j + 1$
    end
    $c_{ii} := \sqrt{\sigma_{ii} - \sum_{j=1}^{i-1} c_{ij}^2}$
    $i := i + 1$
  end

Remark: Since $\mathbf{C}$ is a triangular matrix, it is more efficient to compute $\mathbf{C}\boldsymbol{\xi}$, compared with non-triangular matrices.

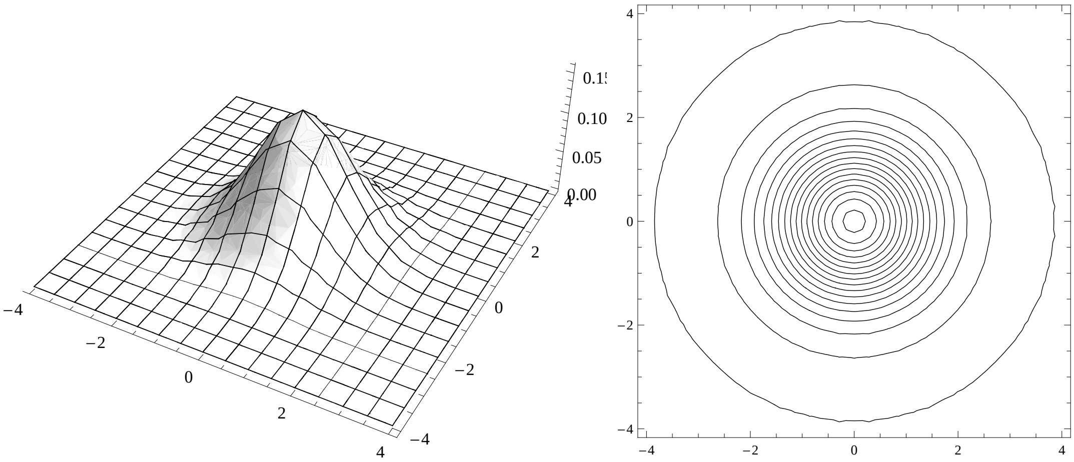

Figure 2.3: The density function of $N_2(\mathbf{0}, \mathbf{I})$ and its contours. Contours are concentric circles (t.j. 2-spheres).

2.2.2 Uniform Distribution on the Unit n-Sphere and in the Unit n-Ball

[4, sec.3.29, p.234] Let $\boldsymbol{\xi} = (\xi_1, ..., \xi_n)'$ be uniformly distributed on the unit n-sphere

$$\mathcal{S}_n = \{\mathbf{x} \in \mathbb{R}^n : x_1^2 + ... + x_n^2 = 1\}.$$

Therefore, the density function of this vector is

$$f(\mathbf{x}) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}}, \ \mathbf{x} \in \mathcal{S}_n,$$

which is reciprocal value of the surface of the unit n-sphere. Since $\boldsymbol{\xi}$ has unit length and uniformly distributed direction, we can see the relationship with the normal distribution. Standard normal distribution is radial symmetric (see fig. 2.3). So, if $\boldsymbol{\eta} \sim N_n(\mathbf{0}, \mathbf{I})$ then it has uniformly distributed direction, therefore we exploit this property to generate from $U_{\mathcal{S}_n}$.

Theorem 2.2.4. *Let* $\boldsymbol{\xi} \sim N_n(\mathbf{0}, \mathbf{I})$. *Then*

$$\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \sim U_{\mathcal{S}_n}.$$

Remark: Since $\boldsymbol{\xi}/\|\boldsymbol{\xi}\| \sim U_{\mathcal{S}_n}$, then $r\boldsymbol{\xi}/\|\boldsymbol{\xi}\| \sim U_{\mathcal{S}_n(r)}$

Now we show, how to generate uniformly on the unit n-ball. Let $\eta \sim U_{(0,1)}^{\frac{1}{n}}$ be the distribution of radius. Then the density function of η is

$$f_\eta(y) = ny^{n-1}, \ 0 \leq y \leq 1.$$

The surface of n-sphere with radius $0 < y < 1$ is

$$f(\mathbf{x}|y) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} y^{n-1}},$$

so the joint distribution function is

$$f(\mathbf{x}|y)f_\eta(y) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}y^{n-1}}ny^{n-1} = \frac{\Gamma(\frac{n}{2})\frac{n}{2}}{\pi^{\frac{n}{2}}} = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}},$$

and that is the volume of the unit n-ball $\mathcal{B}_n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$.

Theorem 2.2.5. *Let $U \sim U_{(0,1)}$, $\boldsymbol{\xi} \sim N_n(\mathbf{0}, \mathbf{I})$ be independent. Then*

$$U^{\frac{1}{n}}\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \sim U_{\mathcal{B}_n}.$$

Remark: Similarly as for the uniform distribution on the sphere, it holds $rU^{\frac{1}{n}}\boldsymbol{\xi}/\|\boldsymbol{\xi}\| \sim U_{\mathcal{B}_n(r)}$.

Chapter 3

$\mathcal{B}^B$ Distribution Family and its Applications

The content of this chapter is based on [7].

3.1 Uniform Distribution on the Unit n-Sphere and its Projection to a Linear Space

Theorem 3.1.1. *Let $\boldsymbol{\nu}$ be a random vector uniformly distributed on the unit n-sphere Then $\|\widetilde{\boldsymbol{\nu}}\|^2 \sim B_{\frac{m}{2},\frac{n-m}{2}}$, where $\widetilde{\boldsymbol{\nu}}$ is a projection of the vector $\boldsymbol{\nu}$ to a m-dimensional linear space.*

Proof. Proof is divided into two parts. First, we prove the theorem for spaces generated by m columns $\mathbf{e}_i$ of $\mathbf{I}_n$. Then, using a suitable transformation, we generalize the theorem for any linear subspace.

Let $\boldsymbol{\xi} \sim N_n(\mathbf{0}, \mathbf{I})$. Then ([4, sec.3.29, p.234])

$$\boldsymbol{\nu} = \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \sim U_{\mathcal{S}_n},$$

where $\mathcal{S}_n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$ is the unit n-sphere. Denote $\mathfrak{N} = \{1, 2, ..., n\}$. To project to a subspace $\mathcal{V}$ generated by m columns of $\mathbf{I}_n$, whose indexes compose set $\mathfrak{M} \subset \mathfrak{N}$, is to keep any of m components and zero the rest $n-m$ components. Denote $\frac{\widetilde{\boldsymbol{\xi}}}{\|\boldsymbol{\xi}\|}$ projection $\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}$ to $\mathcal{V}$. The squared norm of $\frac{\widetilde{\boldsymbol{\xi}}}{\|\boldsymbol{\xi}\|}$ satisfies

$$\left\| \frac{\widetilde{\boldsymbol{\xi}}}{\|\boldsymbol{\xi}\|} \right\|^2 = \frac{\|\widetilde{\boldsymbol{\xi}}\|^2}{\|\boldsymbol{\xi}\|^2} = \frac{\sum_{i\in\mathfrak{M}} \xi_i^2}{\sum_{i\in\mathfrak{N}} \xi_i^2} = \frac{\sum_{i\in\mathfrak{M}} \xi_i^2}{\sum_{i\in\mathfrak{M}} \xi_i^2 + \sum_{i\in\mathfrak{N}-\mathfrak{M}} \xi_i^2}. \tag{3.1}$$

Since $\boldsymbol{\xi} \sim N_n(\mathbf{0}, \mathbf{I})$, then $\xi_i \sim N(0, 1)$ and all components are independent. Thus $\eta_1 = \sum_{i\in\mathfrak{N}-\mathfrak{M}} \xi_i^2 \sim \chi^2_{n-m}$ and $\eta_2 = \sum_{i\in\mathfrak{M}} \xi_i^2 \sim \chi^2_m$. Furthermore, η_1 and η_2 are

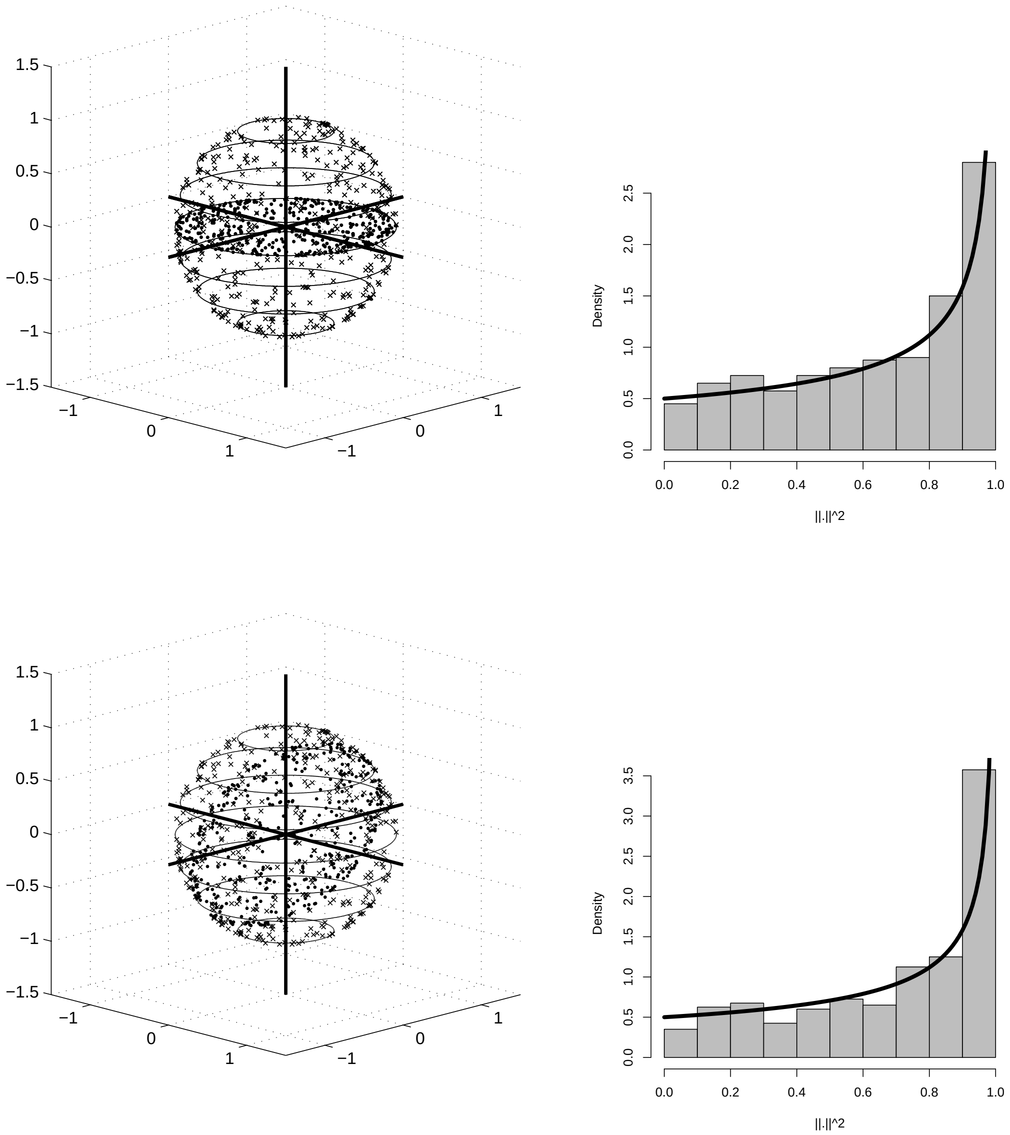

Figure 3.1: Realizations of uniform distribution on the unit sphere (crosses), their projections to different 2-dimensional linear spaces (dots) and the corresponding histograms of squared norm of the projections with inserted density function of $B_{1,\frac{1}{2}}$

independent, because are composed of disjoint sets of $N(0,1)$ distributed random variables. Applying Lemma 1.7.1 to (3.1) we get

$$\|\widetilde{\boldsymbol{\nu}}\|^2 = \left\| \frac{\widetilde{\boldsymbol{\xi}}}{\|\boldsymbol{\xi}\|} \right\|^2 \sim B_{\frac{m}{2},\frac{n-m}{2}}$$

Note that the distribution depends only on dimensions of sphere and linear space, respectively, and not on the values of components.

Now, we generalize the first part of the proof. Let $\mathbf{Q}$ be an orthogonal matrix, then

$$\|\widetilde{\boldsymbol{\nu}}\|^2 \sim B_{\frac{m}{2},\frac{n-m}{2}} \Leftrightarrow \|\mathbf{Q}\widetilde{\boldsymbol{\nu}}\|^2 \sim B_{\frac{m}{2},\frac{n-m}{2}}. \tag{3.2}$$

Denote $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m)$ vectors generating $\mathcal{V}$. Without loss of generality suppose $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are orthonormal and linearly independent, otherwise we use Gram-Schmidt's orthonomalization process and drop the dependent vectors. We have to find such $\mathbf{Q}$, that

$$\mathbf{I}_{m\times n} = \mathbf{Q}\mathbf{V}.$$

Since $\mathbf{Q}^{-1} = \mathbf{Q}'$, we have

$$\mathbf{Q}'\mathbf{I}_{m\times n} = \mathbf{V},$$

i.e., the first m columns of $\mathbf{Q}'$ are composed of $\mathbf{V}$. Complete the matrix $\mathbf{Q}'$ to full basis a transposition we get required transformation matrix. It is (geometrically) obvious that the projection to $\mathcal{V}$ and subsequent transformation by $\mathbf{Q}$ is equivalent to the transformation by $\mathbf{Q}$ and consequent projection to $\mathbf{Q}\mathcal{V} = \text{span}(\mathbf{I}_{m\times n})$, because it is just "rotating the problem". So $\|\widetilde{\boldsymbol{\nu}}\|^2 = \|\mathbf{Q}\widetilde{\boldsymbol{\nu}}_{\mathcal{V}}\|^2 = \|\widetilde{\mathbf{Q}\boldsymbol{\nu}_{\mathbf{I}}}\|^2$.

According to the first part it holds $\|\widetilde{\mathbf{Q}\boldsymbol{\nu}}\|^2 \sim B_{\frac{m}{2},\frac{n-m}{2}}$, so $\|\widetilde{\boldsymbol{\nu}}\|^2 \sim B_{\frac{m}{2},\frac{n-m}{2}}$. □

Lemma 3.1.2. *Let r, m, $n \in \mathbb{N}$, $0 < r < m < n$, $\boldsymbol{\nu}$ be uniformly distributed on the unit n-sphere and $\mathcal{M}$ and $\mathcal{R}$ be an m- and r-dimensional linear space, respectively, $\mathcal{R} \subset \mathcal{M}$. Then*

$$\left\| \widetilde{(\widetilde{\boldsymbol{\nu}}_{\mathcal{M}})_{\mathcal{R}}} \right\|^2 \sim B_{\frac{r}{2},\frac{n-r}{2}}.$$

Proof. A Projection of a vector to $\mathcal{M}$ and consequently to $\mathcal{R}$ gives projection directly to $\mathcal{R}$, i.e. $\widetilde{(\widetilde{\boldsymbol{\nu}}_{\mathcal{M}})_{\mathcal{R}}} = \widetilde{\boldsymbol{\nu}}_{\mathcal{R}}$, thus, $\|\widetilde{(\widetilde{\boldsymbol{\nu}}_{\mathcal{M}})_{\mathcal{R}}}\|^2 = \|\widetilde{\boldsymbol{\nu}}_{\mathcal{R}}\|^2 \sim B_{\frac{r}{2},\frac{n-r}{2}}$. □

Theorem 3.1.3. *Let $r, m, n \in \mathbb{N}$, $0 < r < m < n$ and $\beta_1 \sim B_{\frac{m}{2},\frac{n-m}{2}}$, $\beta_2 \sim B_{\frac{r}{2},\frac{m-r}{2}}$ be independent. Then the random variable $\nu = \beta_1.\beta_2$ satisfies*

$$\nu \sim B_{\frac{r}{2},\frac{n-r}{2}}.$$

Proof. After projecting $\boldsymbol{\xi}$ uniformly distributed on the unit n-sphere to an m-dimensional linear space $\mathcal{M}$ we get $\boldsymbol{\vartheta} = \widetilde{\boldsymbol{\xi}}_{\mathcal{M}}$, a vector with uniformly distributed direction and $\sqrt{B_{\frac{m}{2},\frac{n-m}{2}}}$ distributed norm. We can write

$$\boldsymbol{\vartheta} = \zeta.\boldsymbol{\eta},\ \|\boldsymbol{\eta}\| = 1,\ \boldsymbol{\eta} \sim U_{\mathcal{S}_{\mathcal{M}}},\ \zeta \sim \sqrt{B_{\frac{m}{2},\frac{n-m}{2}}}, \tag{3.3}$$

where $\mathcal{S}_{\mathcal{M}}$ is the unit m-sphere in $\mathcal{M}$.

Let $\mathcal{R}$ be an r-dimensional space. Following the Lemma 3.1.2 we have

$$\widetilde{\boldsymbol{\xi}}_{\mathcal{R}} = \widetilde{\boldsymbol{\vartheta}}_{\mathcal{R}} = \zeta.\widetilde{\boldsymbol{\eta}}_{\mathcal{R}}.$$

Theorem 3.1.1 gives $\nu = \|\widetilde{\boldsymbol{\xi}}_{\mathcal{R}}\|^2 \sim B_{\frac{r}{2},\frac{n-r}{2}}$ a $\nu = \|\zeta.\widetilde{\boldsymbol{\eta}}_{\mathcal{R}}\|^2 = \zeta^2\|\widetilde{\boldsymbol{\eta}}_{\mathcal{R}}\|^2$, where $\zeta^2 \sim B_{\frac{m}{2},\frac{n-m}{2}}$ a $\|\widetilde{\boldsymbol{\eta}}_{\mathcal{R}}\|^2 \sim B_{\frac{r}{2},\frac{m-r}{2}}$. □

We remark that Theorem 1.7.2 at the page 8 is the generalization of this particular case.

3.2 $\mathcal{B}^B$ Family of Random Vectors

We establish the following distribution family of random vectors in an n-dimensional linear space $\mathcal{N}$.

Definition 3.2.1. *Let $\boldsymbol{\nu}$ be a random vector in an n-dimensional linear space $\mathcal{N}$ having the form*

$$\boldsymbol{\nu} = r\zeta.\boldsymbol{\xi},\ \zeta \sim \sqrt{B_{\frac{n}{2},\frac{d}{2}}},\ \boldsymbol{\xi} \sim U_{\mathcal{S}_{\mathcal{N}}},\ d \geq 0,$$

ζ, $\boldsymbol{\xi}$ are independent and $r > 0$ real. Then we denote

$$\boldsymbol{\nu} \sim \mathcal{B}_n^B(r, d),$$

and say that $\boldsymbol{\nu}$ has n-dimensional $\mathcal{B}^B$ distribution with radius r and parameter d.

Remark: As shown below, the from is derived from the beta-product property. The parameter r is more technical and enables us to generalize the family to the ball with radius r.

Let $\boldsymbol{\nu} \sim \mathcal{B}_n^B(r, d)$, according to (3.3) at the page 3.3 it is obvious that it is the projection of a vector uniformly distributed on the $(n + d)$-sphere with radius r. The projection $\widetilde{\boldsymbol{\nu}}_{\mathcal{M}}$ of $\boldsymbol{\nu}$ to an m-dimensional linear space $\mathcal{M}$ (i.e. assembling the projections, since $\boldsymbol{\nu}$ represents a projection) has the form

$$\widetilde{\boldsymbol{\nu}}_{\mathcal{M}} = r\zeta.\eta.\boldsymbol{\xi}_{\mathcal{M}},\ \eta \sim \sqrt{B_{\frac{m}{2},\frac{n-m}{2}}},\ \boldsymbol{\xi}_{\mathcal{M}} \sim U_{\mathcal{S}_{\mathcal{M}}},$$

what is (using Theorem 1.7.2 at the page 1.7.2). equivalent with

$$\widetilde{\boldsymbol{\nu}}_{\mathcal{M}} = r\zeta_{\mathcal{M}}.\boldsymbol{\xi}_{\mathcal{M}},\ \zeta_{\mathcal{M}} \sim \sqrt{B_{\frac{m}{2},\frac{\tilde{d}_{\mathcal{M}}}{2}}},\ \boldsymbol{\xi}_{\mathcal{M}} \sim U_{\mathcal{S}_{\mathcal{M}}},$$

where $\widetilde{d}_{\mathcal{M}} = d + (n - m) \geq 0$, so we can write

$$\widetilde{\boldsymbol{\nu}} \sim \mathcal{B}_m^B(r, \widetilde{d}_{\mathcal{M}}).$$

This shows that $\boldsymbol{\nu}$ and its projection $\widetilde{\boldsymbol{\nu}}_{\mathcal{M}}$ belong to the same distribution family.

Theorem 3.2.2. *The distribution family $\mathcal{B}_n^B(r,d)$ of random vectors is closed with respect to the projection to any m-dimensional ($m \leq n$) linear subspace. After the projection, the second parameter is equal to $d+n-m$.*

Lemma 3.2.3. *Let $\boldsymbol{\nu} \sim \mathcal{B}_n^B(r,d)$. Then $(\nu_{i_1},...,\nu_{i_m})' \sim \mathcal{B}_m^B(r,d+n-m)$.*

Theorem 3.2.4. *The probability density function of $\mathcal{B}_n^B(r,d)$ distribution is*

$$f_{\mathcal{B}_n^B}(\mathbf{x};r,d) = \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}} r^n B(\frac{n}{2},\frac{d}{2})} \left(1 - \frac{\|\mathbf{x}\|^2}{r^2}\right)^{\frac{d}{2}-1}, \ 0 \leq \|\mathbf{x}\| \leq r,$$

which is equal to

$$f_{\mathcal{B}_n^B}(\mathbf{x};r,d) = \frac{\Gamma(\frac{n+d}{2})}{\pi^{\frac{n}{2}} r^n \Gamma(\frac{d}{2})} \left(1 - \frac{\|\mathbf{x}\|^2}{r^2}\right)^{\frac{d}{2}-1}, \ 0 \leq \|\mathbf{x}\| \leq r.$$

Proof. Since $\boldsymbol{\nu} \sim \mathcal{B}_n^B(r,d)$ has uniformly distributed direction and $r\sqrt{B_{\frac{n}{2},\frac{d}{2}}}$ distributed norm, contour sets of this distribution must be concentric spheres, i.e. the distribution is radial symmetric. The value of the density function on the sphere with radius ρ is $Kf^*(\rho;r,d)$, where f^* is a function and K is the normalization constant. Hence the density function of $\mathcal{B}_n^B(r,d)$ must have the form

$$f_{\mathcal{B}_n^B}(\mathbf{x}) = Kf^*(\|\mathbf{x}\|;r,d).$$

Our goal is to find f^*. Let $F_{\|\boldsymbol{\nu}\|}(\rho)$ be the probability that a realization of vector $\boldsymbol{\nu}$ is to occur in the ball $\mathcal{B}_n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < \rho\}$. Then

$$F_{\|\boldsymbol{\nu}\|}(\rho) = \Pr(\|\boldsymbol{\nu}\| < \rho), \ \|\boldsymbol{\nu}\| \sim r\sqrt{B_{\frac{n}{2},\frac{d}{2}}}, \tag{3.4}$$

and, on the other hand,

$$F_{\|\boldsymbol{\nu}\|}(\rho) = \int_{\mathcal{B}_n(\rho)} Kf^*(\|\mathbf{x}\|;r,d)\mathrm{d}\mathbf{x}. \tag{3.5}$$

The relationship (3.4) implies

$$f_{\|\boldsymbol{\nu}\|}(\rho) = \frac{\mathrm{d}F_{\|\boldsymbol{\nu}\|}}{\mathrm{d}\rho} = \frac{2}{r^n B(\frac{n}{2},\frac{d}{2})}\rho^{n-1}\left(1 - \frac{\rho^2}{r^2}\right)^{\frac{d}{2}-1}.$$

Together with (3.5) we get that f^* is the solution of the following equation

$$\frac{2}{r^n B(\frac{n}{2},\frac{d}{2})}\rho^{n-1}\left(1 - \frac{\rho^2}{r^2}\right)^{\frac{d}{2}-1} = \frac{\partial}{\partial\rho}\left(\int_{\mathcal{B}_n(\rho)} Kf^*(\|\mathbf{x}\|;r,d)\mathrm{d}\mathbf{x}\right). \tag{3.6}$$

First, we compute the right side of the equation. Using n-dimensional spherical coordinates ([3, p.65]) we get

$$\begin{aligned}
&\int_{\mathcal{B}_n(\rho)} f^*(\|\mathbf{x}\|;r,d)\mathrm{d}\mathbf{x} \\
&= \int_{R=0}^{\rho}\int_{\varphi_1=0}^{\pi}\cdots\int_{\varphi_{n-2}=0}^{\pi}\int_{\theta=0}^{2\pi} f^*(R;r,d)R^{n-1}\prod_{k=1}^{n-2}\sin^k(\varphi_{n-1-k})\mathrm{d}R\mathrm{d}\varphi_1\ldots\mathrm{d}\varphi_{n-2}\mathrm{d}\theta \\
&= \int_{R=0}^{\rho} f^*(R;r,d)R^{n-1}\mathrm{d}R\int_{\varphi_1=0}^{\pi}\cdots\int_{\varphi_{n-2}=0}^{\pi}\int_{\theta=0}^{2\pi}\prod_{k=1}^{n-2}\sin^k(\varphi_{n-1-k})\mathrm{d}\varphi_1\ldots\mathrm{d}\varphi_{n-2}\mathrm{d}\theta \\
&= \underbrace{\left(2\pi\prod_{k=1}^{n-2}\int_0^{\pi}\sin^k(\varphi_{n-1-k})\mathrm{d}\varphi_{n-1-k}\right)}_{S(n)}\int_{R=0}^{\rho} f^*(R;r,d)R^{n-1}\mathrm{d}R
\end{aligned}$$

Since it is the function of the upper boundary, then

$$\frac{\partial}{\partial\rho}\left(S(n)\int_{R=0}^{\rho} f^*(R;r,d)R^{n-1}\mathrm{d}R\right) = S(n)f^*(\rho;r,d)\rho^{n-1}. \tag{3.7}$$

Setting (3.7) to (3.6) we get

$$\frac{2}{r^nB(\frac{n}{2},\frac{d}{2})}\rho^{n-1}\left(1-\frac{\rho^2}{r^2}\right)^{\frac{d}{2}-1} = KS(n)f^*(\rho;r,d)\rho^{n-1},$$

using some transformations

$$\frac{2}{S(n)r^nB(\frac{n}{2},\frac{d}{2})}\left(1-\frac{\rho^2}{r^2}\right)^{\frac{d}{2}-1} = Kf^*(\rho;r,d).$$

Thus

$$f_{\mathcal{B}_n^B}(\mathbf{x};r,d) = \frac{2}{S(n)r^nB(\frac{n}{2},\frac{d}{2})}\left(1-\frac{\|\mathbf{x}\|^2}{r^2}\right)^{\frac{d}{2}-1},\ 0\le\|\mathbf{x}\|\le 1.$$

We need $S(n)$ to be in an acceptable form. It holds

$$\int_0^{\pi}\sin^n(\varphi)\mathrm{d}\varphi = \sqrt{\pi}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)} = \sqrt{\pi}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}. \tag{3.8}$$

Then

$$S(n) = 2\pi\prod_{k=1}^{n-2}\sqrt{\pi}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})} = 2\pi^{\frac{n}{2}}\frac{\Gamma(\frac{2}{2})}{\Gamma(\frac{3}{2})}.\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{2})}\cdots\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}.\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

Setting $S(n)$ to $f_{\mathcal{B}_n^B}$ we have

$$f_{\mathcal{B}_n^B}(\mathbf{x};r,d) = \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}r^nB(\frac{n}{2},\frac{d}{2})}\left(1-\frac{\|\mathbf{x}\|^2}{r^2}\right)^{\frac{d}{2}-1} = \frac{\Gamma(\frac{n+d}{2})}{\pi^{\frac{n}{2}}r^n\Gamma(\frac{d}{2})}\left(1-\frac{\|\mathbf{x}\|^2}{r^2}\right)^{\frac{d}{2}-1},0\le\|\mathbf{x}\|\le 1.$$

□

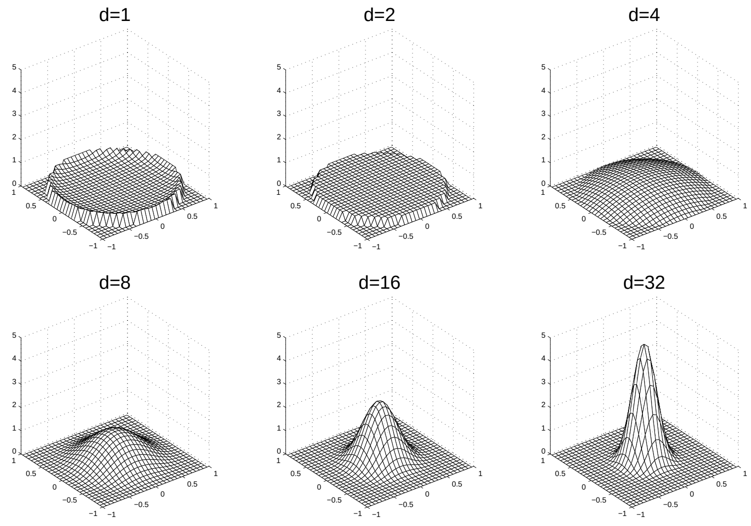

Figure 3.2: Probability density function of $\mathcal{B}^B$ distribution for $n = 2$, $r = 1$ and different values of d.

Two cases of parameter d have an interesting interpretation. If $d = 0$ (limit case of Beta distribution, the density function fails), we project the uniform distribution on the sphere into the original space, i.e. the projection is the uniform distribution on the sphere. It means that $\boldsymbol{\nu} \sim U_{\mathcal{S}_{\mathcal{N}}(r)}$. If $d = 2$, then, according to Lemma 1.7.3, we get

$$\boldsymbol{\nu} \sim r\eta^{\frac{1}{n}}\boldsymbol{\xi},\ \eta \sim U_{(0,1)}, \tag{3.9}$$

i.e. $\boldsymbol{\nu}$ is uniformly distributed on the n-ball with radius r ([4, sec.3.29, p.234]). Note, if we set $d = 2$ into $f_{\mathcal{B}_n^B}$

$$f_{\mathcal{B}_n^B}(\mathbf{x}; r, d = 2) = \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n}{2}} r^n \Gamma(\frac{2}{2})} = \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n}{2}} r^n} = \frac{1}{V_n r^n} = \frac{1}{V_n(r)},$$

where $V_n(r)$ is the volume of the n-ball with radius r.

For parameter $0 < d < 2$ the exponent $\frac{d}{2} - 1$ is negative, thus at the support boundary the value of the density function goes to infinity.

In the figure 3.2 is depicted the density function of $\mathcal{B}_n^B(r, d)$, $n = 2$, $r = 1$, distribution for different values of d. To cope with unboundedness for $d = 1$ we limited the maximal value.

Corollary 3.2.5. *Let $\boldsymbol{\xi} \sim N_{n+k}(\mathbf{0}, \mathbf{I})$ and $\boldsymbol{\nu} = \frac{\widetilde{\boldsymbol{\xi}}}{\|\boldsymbol{\xi}\|}$, where $\widetilde{\boldsymbol{\xi}}$ is a vector composed of n components of $\boldsymbol{\xi}$. Then $\boldsymbol{\nu} \sim \mathcal{B}_n^B(1, k)$. For $k = 2$ we have uniform distribution in the unit ball.*

Remark: Corollary 3.2.5 proposes an efficient algorithm for generating from the interior of n-ball, if we have physical $N(0, 1)$ generator.

Proof. Since $\boldsymbol{\xi}/\|\boldsymbol{\xi}\| \sim U_{\mathcal{S}_{n+k}}$. It is easy to see that $\widetilde{\boldsymbol{\xi}}$ is projection of $n+k$-dimensional vector $\boldsymbol{\xi}$ to n-dimensional linear space span$(\mathbf{e}_1, ..., \mathbf{e}_k)$, thus it is the projection of $\boldsymbol{\xi}/\|\boldsymbol{\xi}\|$. Using Lemma 3.2.3 we prove the corollary. □

Remark: Projecting uniform distribution on the unit 3-sphere to axis O_Z we have the uniform distribution on $(-1, 1)$ (in O_Z direction). Remark that the probability that

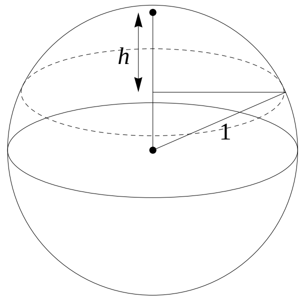

Figure 3.3: Spherical cap.

a random vector occurs on the spherical cap with height h (denote Cap(h)) is the surface of the cap ($S_{\text{Cap}(h)} = 2\pi h$) divided by the surface of the sphere ($S_{\text{Sph}} = 4\pi$), i.e.

$$P(\xi \in \text{Cap}(h)) = \frac{S_{\text{Cap}(h)}}{S_{\text{Sph}}} = \frac{2\pi h}{4\pi} = \frac{h}{2}.$$

On the other hand, the density function of $U_{(-1,1)}$ is $f(x) = 1/2$, $x \in (-1, 1)$ and the probability that a realization occurs in the interval $I = (1-h, 1)$ of length h is $h/2$, too.

We also have, that the surface of the cap is proportional to its height, i.e

$$S_{\text{Cap}(h)} = h\frac{S_{\text{Sph}}}{2}$$

Below we describe some basic properties of $\mathcal{B}_n^B(r, d)$ distribution.

Theorem 3.2.6. *Let* $\boldsymbol{\nu} \sim \mathcal{B}_n^B(r, d)$. *Then*

1. $\text{E}[\boldsymbol{\nu}] = \mathbf{0}$,

2. *components* ν_i, $i = 1, ..., n$, *are dependent and* $\text{Cov}[\nu_i, \nu_j] = 0$ *for* $i \neq j$,

3. $\text{Var}[\boldsymbol{\nu}] = \frac{r^2}{n+d}\mathbf{I}$.

Proof. 1 is true because the distribution is radial symmetric.

The radial symmetry implies that any permutation of components gives the same random vector. Since $\text{E}[\nu_i] = 0$ it holds $\text{Cov}[\nu_i, \nu_j] = \text{E}[\nu_i \nu_j]$, $i \neq j$. If 2 holds for the first and the second component, then radial symmetry implies that it holds for any two components. Thus, to prove 2 we have to show that $\text{E}[\nu_1 \nu_2] = 0$, i.e.

$$0 = \text{Cov}[\nu_1, \nu_2] = \int_{\mathcal{B}_n(\rho)} \nu_1 \nu_2 \frac{\Gamma(\frac{n+d}{2})}{\pi^{\frac{n}{2}} r^n \Gamma(\frac{d}{2})} \left(1 - \frac{\|\boldsymbol{\nu}\|^2}{r^2}\right)^{\frac{d}{2}-1} \text{d}\boldsymbol{\nu}.$$

Using n-spherical coordinates we get

$$\begin{aligned}\mathrm{Cov}[\nu_1,\nu_2] &= K\int_0^{\rho} R^{n+1}\left(1-\frac{R^2}{r^2}\right)^{\frac{d}{2}-1}\mathrm{d}R\prod_{k=1}^{n-4}\int_0^{\pi}\sin^k(\varphi_{n-1-k})\mathrm{d}\varphi_{n-1-k}\\ &\quad \underbrace{\int_0^{\pi}\sin^{n-1}(\varphi_1)\cos(\varphi_1)\mathrm{d}\varphi_1}_{=0,\text{ for } n>0}\int_0^{\pi}\sin^{n-3}(\varphi_2)\cos(\varphi_2)\mathrm{d}\varphi_2\\ &= 0\end{aligned}$$

Since $\mathcal{B}_n^B(r,d) \overset{d}{\not\sim} N_n(\mathbf{0},\sigma^2(n,r,d)\mathbf{I})$, due to Herschel-Maxwell theorem, the distribution must have dependent components.

We prove 3 in the following way. The radial symmetry and zero covariance of the components give $\mathrm{Var}[\boldsymbol{\nu}] = \sigma^2(n,r,d)\mathbf{I}$, i.e. all components have the same variance $\sigma^2(n,r,d)$ that depends on n, r and d. Projecting $\boldsymbol{\nu}$ to a one-dimensional space we get random variable

$$\begin{aligned}\xi &\sim r\sqrt{B_{\frac{1}{2},\frac{d+n-1}{2}}} \overset{d}{\sim} \mathcal{B}_1^B(r,d+n-1),\\ \xi^2 &\sim r^2 B_{\frac{1}{2},\frac{d+n-1}{2}}.\end{aligned}$$

Since $\mathrm{E}[\xi]=0$, then $\mathrm{Var}[\xi]=\mathrm{E}[\xi^2]$ is the variance of one component. Using the mean of the Beta distribution we get

$$\sigma^2(n,r,d) = \mathrm{Var}[\xi] = \mathrm{E}[\xi^2] = r^2\frac{\frac{1}{2}}{\frac{1}{2}+\frac{d+n-1}{2}} = \frac{r^2}{n+d}.$$

□

Corollary 3.2.7. *Let $r>0$ be fixed. Then for $d\to\infty$ the $\mathcal{B}_n^B(r,d)$ distribution converges to Dirac distribution in $\mathbf{0}\in\mathbb{R}^n$.*

Theorem 3.2.8. *Let $\boldsymbol{\nu}\sim\mathcal{B}_n^B(r,d)$, $\boldsymbol{\nu}=(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)'$, and $\boldsymbol{\nu}_1,\mathbf{x}\in\mathbb{R}^m$, $\boldsymbol{\nu}_2,\mathbf{y}\in\mathbb{R}^{n-m}$. Then*

$$\Pr(\boldsymbol{\nu}_1<\mathbf{x}|\boldsymbol{\nu}_2=\mathbf{y}) = \int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_m} f_{\mathcal{B}_n^B}(\boldsymbol{\xi};r,d|\mathbf{y})\mathrm{d}\boldsymbol{\xi},$$

where

$$f_{\mathcal{B}_n^B}(\boldsymbol{\xi};r,d|\mathbf{y}) = \frac{\Gamma(\frac{m}{2})}{\pi^{\frac{m}{2}}B(\frac{m}{2},\frac{d}{2})}\cdot\frac{\left(1-\frac{\|\mathbf{y}\|^2}{r^2}-\frac{\|\boldsymbol{\xi}\|^2}{r^2}\right)^{\frac{d}{2}-1}}{\left(1-\frac{\|\mathbf{y}\|^2}{r^2}\right)^{\frac{d+m}{2}-1}}.$$

Proof. According to [13, sec.1.3, p.13], for the conditional density function it holds

$$f_{\mathcal{B}_n^B}(\mathbf{x};r,d|\mathbf{y}) = \frac{f_{\mathcal{B}_n^B}((\mathbf{x},\mathbf{y});r,d)}{\int_{\mathcal{B}_m(1-\|\mathbf{y}\|^2)} f_{\mathcal{B}_n^B}((\mathbf{u},\mathbf{y});r,d)\mathrm{d}\mathbf{u}}.$$

Since the denominator stands for the marginal density funcion of the last $n-m$ components, we use projection to reduce m dimensions. Therefore

$$
\begin{aligned}
f_{\mathcal{B}_n^B}(\mathbf{x}; r, d|\mathbf{y}) &= \frac{\frac{\Gamma(\frac{n+d}{2})}{\pi^{\frac{n}{2}} r^n \Gamma(\frac{d}{2})}\left(1-\frac{\|\mathbf{x}\|^2}{r^2}-\frac{\|\mathbf{y}\|^2}{r^2}\right)^{\frac{d}{2}-1}}{\frac{\Gamma(\frac{n-m+m+d}{2})}{\pi^{\frac{n-m}{2}} r^{n-m}\Gamma(\frac{d+m}{2})}\left(1-\frac{\|\mathbf{y}\|^2}{r^2}\right)^{\frac{d+m}{2}-1}} \\
&= \frac{\Gamma(\frac{d+m}{2})}{\pi^{\frac{m}{2}} r^m \Gamma(\frac{d}{2})} \cdot \frac{\left(1-\frac{\|\mathbf{x}\|^2}{r^2}-\frac{\|\mathbf{y}\|^2}{r^2}\right)^{\frac{d}{2}-1}}{\left(1-\frac{\|\mathbf{y}\|^2}{r^2}\right)^{\frac{d+m}{2}-1}}.
\end{aligned}
$$

□

In the following theorem we show that conditional distribution of $\mathcal{B}_n^B(r,d)$ belongs into the same family.

Theorem 3.2.9. *Suppose* $\mathbf{x}$, $\mathbf{y}$, $\boldsymbol{\nu}$ *satisfy the assumptions in Theorem 3.2.8,* $\boldsymbol{\nu}_2 = \mathbf{y}$ *and* $\|\mathbf{y}\| < r$*. Then*

$$\boldsymbol{\nu}_1 \sim \mathcal{B}_m^B\left(\sqrt{r^2 - \|\mathbf{y}\|^2}, d\right).$$

Proof. The density function of $\boldsymbol{\nu}_1$ is $f_{\mathcal{B}_n^B}(\mathbf{x}; r, d|\mathbf{y})$. If $\|\mathbf{y}\| < r$, then $\frac{1}{\sqrt{1-\frac{\|\mathbf{y}\|^2}{r^2}}}$ is defined, and the density function of $\frac{1}{\sqrt{1-\frac{\|\mathbf{y}\|^2}{r^2}}}\boldsymbol{\nu}_1$ is

$$
\begin{aligned}
f(\mathbf{x}; \mathbf{y}, d) &= \left(\sqrt{1-\frac{\|\mathbf{y}\|^2}{r^2}}\right)^m f_{\mathcal{B}_n^B}\left(\sqrt{1-\frac{\|\mathbf{y}\|^2}{r^2}}\mathbf{x}; r, d|\mathbf{y}\right) \\
&= \left(1-\frac{\|\mathbf{y}\|^2}{r^2}\right)^{\frac{m}{2}} \frac{\Gamma(\frac{m}{2})}{\pi^{\frac{m}{2}} r^m B(\frac{m}{2}, \frac{d}{2})} \cdot \frac{\left(1-\frac{\|\mathbf{y}\|^2}{r^2}\right)^{\frac{d}{2}-1}\left(1-\frac{\|\mathbf{x}\|^2}{r^2}\right)^{\frac{d}{2}-1}}{\left(1-\frac{\|\mathbf{y}\|^2}{r^2}\right)^{\frac{d+m}{2}-1}} \\
&= \frac{\Gamma(\frac{m}{2})}{\pi^{\frac{m}{2}} r^m B(\frac{m}{2}, \frac{d}{2})}\left(1-\frac{\|\mathbf{x}\|^2}{r^2}\right)^{\frac{d}{2}-1}.
\end{aligned}
$$

So $\frac{1}{\sqrt{1-\frac{\|\mathbf{y}\|^2}{r^2}}}\boldsymbol{\nu}_1 \sim \mathcal{B}_m^B(r,d)$, and we have $\boldsymbol{\nu}_1 \sim \mathcal{B}_m^B\left(r\sqrt{1-\frac{\|\mathbf{y}\|^2}{r^2}}, d\right)$. □

The shape of the density function of the $\mathcal{B}^B$ for greater values of the parameter d is similar to the density of normal distribution. Let $\boldsymbol{\nu} \sim \mathcal{B}_n^B(\sqrt{n+d}, d)$, then $\mathrm{E}[\boldsymbol{\nu}] = \mathbf{0}$ and $\mathrm{Var}[\boldsymbol{\nu}] = \mathbf{I}_n$. We show, that for $d \to \infty$, $\mathcal{B}^B$ distribution converges to normal distribution.

Theorem 3.2.10. *Let* $\boldsymbol{\nu} \sim \mathcal{B}_n^B(\sqrt{n+d}, d)$*. Then for* $d \to \infty$*,* $\boldsymbol{\nu}$ *has* $N_n(\mathbf{0}, \mathbf{I})$ *distribution.*

Proof. We show, that for $d \to \infty$, the density function of $\mathcal{B}_n^B(\sqrt{n+d}, d)$ converges to the density function of $N_n(\mathbf{0}, \mathbf{I})$ distribution.

$$\begin{aligned}
\lim_{d\to\infty} f_{\mathcal{B}_n^B}(\mathbf{x}; r, d) &= \lim_{d\to\infty} \frac{\Gamma(\frac{n+d}{2})}{\pi^{\frac{n}{2}}(n+d)^{\frac{n}{2}}\Gamma(\frac{d}{2})}\left(1-\frac{\|\mathbf{x}\|^2}{n+d}\right)^{\frac{d}{2}-1} \\
&= \lim_{d\to\infty} \frac{\Gamma(\frac{n+d}{2})}{(2\pi)^{\frac{n}{2}}(\frac{n+d}{2})^{\frac{n}{2}}\Gamma(\frac{d}{2})}\left(1-\frac{\|\mathbf{x}\|^2/2}{(n+d)/2}\right)^{\frac{n+d}{2}-\frac{n}{2}-1} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\frac{\|\mathbf{x}\|^2}{2}} \lim_{d\to\infty} \frac{\Gamma(\frac{n+d}{2})}{(\frac{n+d}{2})^{\frac{n}{2}}\Gamma(\frac{d}{2})}.
\end{aligned}$$

For n fixed, if $d \to \infty$, then $\frac{n+d}{2} \to \infty$. So

$$\lim_{d\to\infty} \frac{\Gamma(\frac{n+d}{2})}{(\frac{n+d}{2})^{\frac{n}{2}}\Gamma(\frac{d}{2})} = \lim_{\frac{n+d}{2}\to\infty} \frac{\Gamma(\frac{n+d}{2})}{(\frac{n+d}{2})^{\frac{n}{2}}\Gamma(\frac{d}{2})} = \lim_{N\to\infty} \frac{\Gamma(N)}{N^{\frac{n}{2}}\Gamma(N-\frac{n}{2})}.$$

Using [1, 6.1.46, p.257] we get the limit equal 1. □

Theorem 3.2.11. *Let $d > 0$ be fixed and $\{\boldsymbol{\xi}_n\}_{n=k}^{\infty}$, $\boldsymbol{\xi}_n \sim \mathcal{B}_n^B(\sqrt{n+d}, d)$ be a sequence of random vectors. Let $\{\boldsymbol{\nu}_n\}_{n=k}^{\infty}$ be such sequence of random vectors that $\boldsymbol{\nu}_n = (\boldsymbol{\xi}_n^{(1)}, ..., \boldsymbol{\xi}_n^{(k)})'$, where $\boldsymbol{\xi}_n^{(i)}$ is ith component of vector $\boldsymbol{\xi}_n$. Then $\boldsymbol{\nu}_n$ has k-dimensional normal distribution as $n \to \infty$.*

Proof. Let $B \sim B_{\frac{n}{2},\frac{d}{2}}$ and $\boldsymbol{\eta} \sim N_n(\mathbf{0}, \mathbf{I})$. Then by the definition of the distribution $\mathcal{B}^B$ we have

$$\boldsymbol{\xi}_n = \sqrt{n+d}\sqrt{B}\frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}$$

It is easy to see, that $\|\boldsymbol{\eta}\|^2 \sim \chi_n^2$, thus $\|\boldsymbol{\eta}\|^2/(n+d) \xrightarrow{P} 1$ as $n \to \infty$. Using Lemma 1.7.1 at the page 7 and using the same asymptotic we get $B \xrightarrow{P} 1$ as $n \to \infty$. □

3.3 A Class of Algorithms for Generating Random Vectors on the Surface and in the Interior of the Unit n-Ball

Let $\boldsymbol{\nu}_m^k = (\nu_k, ..., \nu_{k+m-1})'$ be an m-dimensional subvector of $\boldsymbol{\nu} \sim \mathcal{B}_n^B(r, d)$ and $\mathbf{y}_{1,k-1}$ be a realization of the first $k-1$ components of vector $\boldsymbol{\nu}$. Then using Lemma 3.2.3, Theorem 3.2.2 and Theorem 3.2.9 we get the conditional distribution

$$\boldsymbol{\nu}_m^k \sim \mathcal{B}_m^B\left(\sqrt{r^2 - \|\mathbf{y}_{1,k-1}\|^2}, d+n-m-(k-1)\right). \tag{3.10}$$

Let $n_1, n_2, ..., n_j,\ n_1 + n_2 + ... + n_j = n$, be such we can we can generate from $\mathcal{B}_{n_i}^B$, $i = 1, ..., j$, easily. Then we can generate components sequentially. An easy-to-generate sequence we get if we choose, for example, $n_i = 2$ for all i. This case is described in the next subsection. Then we describe more "exotic" cases.

3.3.1 Application: Simulating Uniformly on the Unit n-Sphere and n-Ball

Here we discuss a special case of our class of algorithms, when we generate components in pairs. First we analyze generating on the surface on the n-ball.

If we generate components ν_{2i-1}, ν_{2i}, $i = 1, 2, ..., n/2$, simultaneously, then we have

$$(\nu_{2i-1}, \nu_{2i})' \sim \mathcal{B}_2^B \left(\sqrt{1 - \|\mathbf{y}_{1,2i-2}\|^2}, n - 2i \right), \tag{3.11}$$

what is easy to generate, since

$$\|(\nu_{2i-1}, \nu_{2i})'\| \sim \sqrt{(1 - \|\mathbf{y}_{1,2i-2}\|^2)(1 - U^{\frac{2}{n-2i}})},$$

and for the random direction we use the rejection method for the unit circle. This enables us to generate uniformly on the unit n-sphere without using trigonometric functions, exponentials or logarithms.

Remark: This algorithm was described by Marsaglia (see [11]) up to dimension 4. Yang et al. (see [17]) generalized Marsaglia's method for any dimension. Note that it is only a special case of our class of algorithms.

Remark: Since

$$\frac{\pi^{\frac{n}{2}}}{2^n \Gamma(\frac{n}{2} + 1)} = \frac{(\pi/4)^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} < \frac{0.8^{\frac{n}{2}}}{\lfloor \frac{n}{2} \rfloor !} \searrow 0, \text{ for } n \to \infty$$

is the probability of successful realization from uniform distribution in the unit n-ball, if we generate from n-cube, the rejection method is getting useless with increasing dimension.

Algorithm 3.3.1. *This algorithm generates a point* $\mathbf{X}$ *uniformly distributed on the unit n-sphere without using trigonometric functions, exponentials or logarithms.*

function: $[X] = U\_nSphere(n)$
$\quad \rho := 1$
$\quad$ ***for*** $i = 1 : \lfloor \frac{n+1}{2} \rfloor - 1$
$\quad$ *P1:* $\quad$ *generate* $A, B \overset{iid}{\sim} U_{(-1,1)}$
$\quad\quad R^2 := A^2 + B^2$
$\quad\quad$ ***if*** $R^2 > 1$ ***goto*** *P1*
$\quad\quad N_1 := \rho(1 - (R^2)^{\frac{2}{n-2i}}),\ N_2 := \sqrt{N_1/R^2}$
$\quad\quad X_{2i-1} := N_2 A,\ X_{2i} := N_2 B$
$\quad\quad \rho := \rho - N_1$
$\quad$ ***end***
$\quad$ ***if*** *n is an odd number*
$\quad\quad$ *generate* A *such that* $\Pr(A = -1) = 0.5, \Pr(A = 1) = 0.5$
$\quad\quad X_n := \sqrt{\rho} * A$

else
P2: *generate* $A, B \overset{iid}{\sim} U_{(-1,1)}$
$R^2 := A^2 + B^2$
if $R^2 > 1$ ***goto*** *P2*
$N := \sqrt{\rho/R^2}$
$X_{n-1} := NA,\ X_n := NB$
end

Now, suppose we want to generate uniformly in the unit n-ball. Applying idea in formula (3.11) for sequential simulation, where the second parameter is equal $n-2i+2$, we get the following algorithm.

Algorithm 3.3.2. *This algorithm generates a point* $\mathbf{X}$ *uniformly distributed in the unit n-ball without using trigonometric functions, exponentials or logarithms.*

function: $[X] = U\_nBall(n)$
$\rho := 1$
for $i = 1 : \lfloor \frac{n+1}{2} \rfloor - 1$
P1: *generate* $A, B \overset{iid}{\sim} U_{(-1,1)}$
$R^2 := A^2 + B^2$
if $R^2 > 1$ ***goto*** *P1*
$N_1 := \rho(1 - (R^2)^{\frac{2}{2+n-2i}}),\ N_2 := \sqrt{N_1/R^2}$
$X_{2i-1} := N_2 A,\ X_{2i} := N_2 B$
$\rho := \rho - N_1$
end
if n *is an odd number*
generate $A \sim U_{(-1,1)}$
$X_n := \sqrt{\rho} * A$
else
P2: *generate* $A, B \overset{iid}{\sim} U_{(-1,1)}$
$R^2 := A^2 + B^2$
if $R^2 > 1$ ***goto*** *P2*
$N := \sqrt{\rho}$
$X_{n-1} := NA,\ X_n := NB$
end

3.3.2 Special Cases

Here we describe some other special cases of our class of algorithms.

Example 3.3.3 (Generating uniformly on the surface of the 4-ball)**.** *To generate on the surface of the unit 4-ball we choose sequential method as follows: first we generate the first two components, then the third and finally the fourth one. Using formula (3.10) we get:*

1. *Generate* $(x_1, x_2)' \sim U_{\mathcal{B}_2}$ *using rejection method*

2. *Generate* $U \sim U_{(0,1)}$ *and set* $x_3 = \sqrt{1 - x_1^2 - x_2^2}\cos(\pi U)$ *(note that* $\mathcal{B}_1^B(1,1)$ *is the arcsine distribution. Using the inverse transform theorem we get the relation for* x_3*)*

3. *Set* $x_4 = \varsigma\sqrt{1 - x_1^2 - x^2 - x_3^2}$*, where* ς *is a random sign*

Then $\mathbf{x}$ *is uniformly distributed on the surface of the unit 4-sphere.*

Example 3.3.4 (Generating uniformly in the interior of the 4-ball)**.** *To generate in the interior of the unit 4-ball we generate the first two components and then we generate the last two components. We have*

1. *Generate* $(y_1, y_2)' \sim U_{\mathcal{B}_2}$ *and* $(y_3, y_4)' \sim U_{\mathcal{B}_2}$ *using rejection method and* $U \sim U_{(0,1)}$

2. *Set* $(x_1, x_2)' = U^{1/4}(y_1, y_2)'$

3. *Set* $(x_3, x_4)' = \sqrt{1 - x_1^2 - x_2^2}(y_3, y_4)'$

Then $\mathbf{x}$ *is uniformly distributed in the interior of the unit 4-sphere.*

Example 3.3.5 (Generating uniformly on the surface of the 5-ball)**.** *To get a random vector uniformly distributed on the surface of the unit 5-ball we generate the first component, then the second and third component simultaneously, then the fourth and the fifth component.*

1. *Generate* U_1*,* $U_2 \sim U_{(0,1)}$ *and a random sign* ς_1 *and set* $x_1 = \varsigma_1 U_1 U_2^{1/3}$ *(the product of the two uniformly distributed random variables comes from formula (1.3) at the page 10)*

2. *Generate* $(y_1, y_2)' \sim U_{\mathcal{B}_2}$ *and set* $(x_2, x_3)' = \sqrt{1 - x_1^2}(y_1, y_2)'$

3. *Generate* $V \sim U_{(0,1)}$ *and set* $x_4 = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}\cos(\pi V)$

4. $x_5 = \varsigma_2\sqrt{1 - x_1^2 - x_2^2 - x_3^2 - x_4^2}$*, where* ς_2 *is a random sign*

Then $\mathbf{x}$ *is uniformly distributed on the surface of the unit 5-sphere.*

Example 3.3.6 (Generating uniformly in the interior of the 5-ball)**.** *To get a random vector uniformly distributed in the interior of the unit 5-ball we generate the first three components and then we generate the last two components. The result is similar to the case of the unit 4-ball.*

1. *Generate* $(y_1, y_2, y_3)' \sim U_{\mathcal{B}_3}$ *and* $(y_4, y_5)' \sim U_{\mathcal{B}_2}$ *using rejection method and* $U \sim U_{(0,1)}$

2. *Set* $(x_1, x_2, x_3)' = U^{1/5}(y_1, y_2, y_3)'$

3. *Set* $(x_4, x_5) = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}(y_4, y_5)'$

Then $\mathbf{x}$ *is uniformly distributed in the interior of the unit 5-sphere.*

Chapter 4

Comparing the Efficiency of the New Generators with Classic Generators from Uniform Distribution on the Unit n-Sphere and in the Unit n-Ball

4.1 Simulation from the Unit n-Sphere

In this section we compare the efficiency of the generator described by Algorithm 3.3.1 (C++ source code available in Subsection 5.4.2) with a classic algorithm

$$\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|},\ \boldsymbol{\xi} \sim N_n(\mathbf{0}, \mathbf{I}), \tag{4.1}$$

where we used Box-Muller generator for simulating standard normal variables (C++ source code available in Subsection 5.4.1). For appropriate comparison we used programming language C++[1]. The Results are exhibited in the figure 4.1 and the corresponding times are available in Table 5.1 in Appendix.

The result shows that the time difference increases linearly with increasing dimension. The new generator needs significantly less simulation time than the classic generator (i.e. based on (4.1) using the Box-Muller method for generating normal variates).

4.2 Simulation from the Unit n-Ball

To simulate uniformly in the unit n-ball a couple of methods are available. We compare the following methods:

[1]We used Bloodshed DevC++ version 4.9.9.2. See *http://www.bloodshed.net* for more information. Application was programmed as Console application.

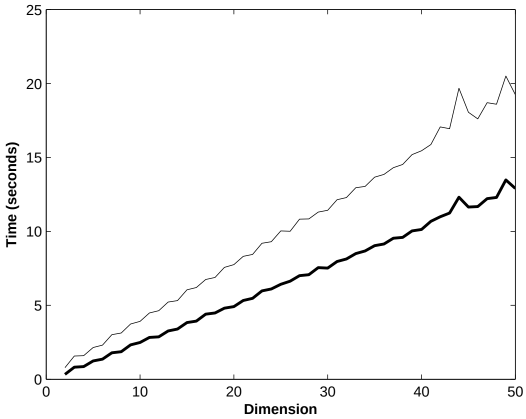

Figure 4.1: Comparing the efficiency of the classic (thin line) and the new (thick line) algorithm for generating uniformly on the unit n-sphere. Horizontal axis represents the dimension of the vector and vertical axis the simulation time of 10^6 vectors. The simulation test is done in C++. Times are available in Table 5.1 in Appendix.

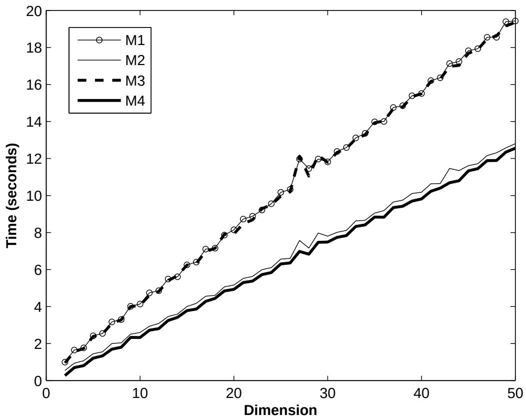

Figure 4.2: Comparing the efficiency of M1–M4 for generating uniformly in the unit n-ball. Horizontal axis represents the dimension of the vector and vertical axis the simulation time of 10^6 vectors. The simulation test is done in C++. Times are available in Table 5.2 in Appendix.

M1 The classic algorithm

$$\boldsymbol{\nu} = U^{1/n}\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \sim U_{\mathcal{B}_n},\ U \sim U_{(0,1)},\ \boldsymbol{\xi} \sim N_n(\mathbf{0}, \mathbf{I}),$$

(C++ source code available in Subsection 5.4.3 and R code in Subsection 5.4.4)

M2 The classic algorithm upgraded using Algorithm 3.3.1

$$\boldsymbol{\nu} = U^{1/n}\boldsymbol{\eta} \sim U_{\mathcal{B}_n},\ U \sim U_{(0,1)},\ \boldsymbol{\eta} \sim U_{\mathcal{S}_n},$$

(C++ source code available in Subsection 5.4.5)

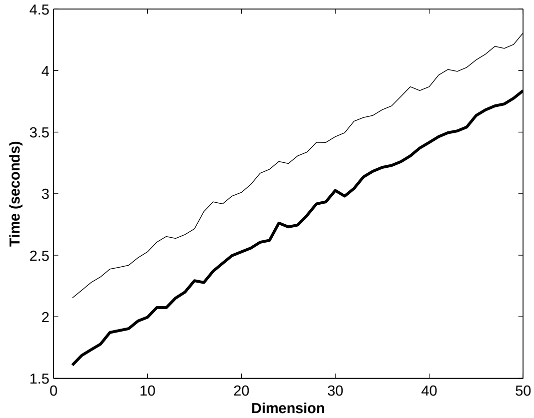

Figure 4.3: Comparing the efficiency of M1 (thin line) and M3 (thick line) algorithm for generating uniformly in the unit n-ball. Horizontal axis represents the dimension of the vector and vertical axis the simulation time of 10^5 vectors. The simulation test is done in R. Times are available Table 5.3 in Appendix.

M3 The new algorithm, a result of Corollary 3.2.5

$$\boldsymbol{\nu} = \frac{\boldsymbol{\xi}}{\sqrt{(\sum_{i=1}^{n} \xi_i^2) + \eta_1^2 + \eta_2^2}} \sim U_{\mathcal{B}_n},\ \boldsymbol{\xi} \sim N_n(\mathbf{0}, \mathbf{I}), \eta_1,\ \eta_2 \sim N(0,1), \quad (4.2)$$

(C++ source code available in Subsection 5.4.6 and R code in Subsection 5.4.7)

M4 The new algorithm described in Algorithm 3.3.2. (C++ source code available in Subsection 5.4.8)

The figure 4.2 is depicts a dependence of the times of simulations for each method for different dimensions (the times are available in Table 5.2 in Appendix). Methods M2 and M4, which use the sequential simulation of the vector's components, are more efficient than M1 and M3, even for higher dimensions. The method M4 has slight, but noticeable advantage over M2.

Now we focus our attention to the methods M1 and M3. Even though the method M1 was as efficient as M3, we compared both methods using the programming language of the R software environment[2]. The difference between C++ and R is that R "takes" a realization from "somewhere" (a pre-compiled function), which stands (in a sense) for a physical $N(0,1)$ generator. The results (see fig. 4.3 and Table 5.3 in Appendix) show that M3 is, from this point of view, more efficient than M1.

[2]We used R version 2.8.1 by The R Foundation for Statistical Computing. See *http://www.r-project.org* for more information.

Chapter 5

Appendix

5.1 n-Spherical Coordinates [3, p.65]

Let be $\mathbf{I}_n$ be given basis, i.e. Cartesian coordinate system, and $\mathbf{x}$ is a vector of norm r with components x_i with respect to this basis. Then

$$\begin{aligned} x_1 &= r\cos(\phi_1), \\ x_j &= r\cos(\psi_j)\prod_{k=1}^{j-1}\sin(\phi_k),\ j=2,...,n-2, \\ x_{n-1} &= r\sin(\theta)\prod_{k=1}^{n-2}\sin(\phi_k), \\ x_n &= r\sin(\theta)\prod_{k=1}^{n-2}\sin(\phi_k), \end{aligned}$$

where $0 \leq \phi_j \leq \pi,\ j=1,...,n-2; 0 \leq \theta < 2\pi; 0 \leq r < \infty$. The Jacobian of this transformation is

$$J = r^{n-1}\prod_{k=1}^{n-2}\sin^k(\phi_{n-1-k}).$$

5.2 Formulas from [1]

[1, 15.1.8, p.556]:

$${}_2F_1(a,b;b;z) = (1-z)^{-a},$$

[1, 15.3.1, p.558]:

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t,\ \Re[c] > \Re[b] > 0,$$

[1, 6.1.46, p.257]:

$$\lim_{n\to\infty} n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)} = 1.$$

5.3 Herschel–Maxwell Characterization of the Normal Distribution

The characterization states, that if a random vector $\boldsymbol{\xi}$ is rotationally invariant (i.e. $\boldsymbol{\xi} \overset{d}{\sim} \mathbf{Q}\boldsymbol{\xi}$, $\mathbf{Q}$ be orthogonal) and components ξ_i, ξ_j are independent, $i \neq j$, then the distribution of $\boldsymbol{\xi}$ is the multidimensional normal with mean $\mathbf{0}$.

5.4 Source Codes

5.4.1 The Classic Method for $U_{\mathcal{S}_n}$ (C++)

```
int i, n=3, k=n/2;
double A, B, R2, NORMAL[n], P, NN=0;
for(i=1;i<=k;i++){
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  P=sqrt(-2*log(R2)/R2);
  NORMAL[2*i-2]=P*A;
  NORMAL[2*i-1]=P*B;
  NN=NN+P*P*(A*A+B*B);
}
if(n%2){
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  P=sqrt(-2*log(R2)/R2)
  NORMAL[n-1]=P*A;
  NN=NN+P*P*A*A;
}
for(i=1;i<=n;i++)
  NORMAL[i-1]=NORMAL[i-1]/sqrt(NN);
```

5.4.2 Algorithm 3.3.1 (C++)

```
int i, n=3, k=(n+1)/2;
double RADIUS=1, A, B, R2, POINT[n], SQ_OF_NORM, P, S;
```

```
for(i=1; i<=k-1; i++){
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  SQ_OF_NORM=RADIUS*(1-pow(R2,2.0/(n-2*i)));
  P=sqrt(SQ_OF_NORM/R2);
  POINT[2*i-2]=P*A;
  POINT[2*i-1]=P*B;
  RADIUS=RADIUS-SQ_OF_NORM;
}
if(n%2){
  S=0.5-drand();
  POINT[n-1]=sqrt(RADIUS)*S/fabs(S);
}
else{
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
    }
  P=sqrt(RADIUS/R2);
  POINT[n-2]=P*A;
  POINT[n-1]=P*B;
}
```

5.4.3 M1 (C++)

```
int i, n=3, k=n/2;
double A, B, R2, T, NORMAL[n], P, NN=0;
T=pow(drand(),1.0/n);
for(i=1;i<=k;i++){
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  P=sqrt(-2*log(R2)/R2);
  NORMAL[2*i-2]=T*P*A;
  NORMAL[2*i-1]=P*B;
  NN=NN+P*P*(A*A+B*B);
```

```
}
if(n%2){
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  P=sqrt(-2*log(R2)/R2);
  NORMAL[n-1]=P*A;
  NN=NN+P*P*A*A;
}
for(i=1;i<=n;i++)
  NORMAL[i-1]=NORMAL[i-1]/sqrt(NN);
```

5.4.4 M1 (R)

```
n<-3
xi<-rnorm(n)
X<-(runif(1))^(1/n)*xi/sqrt(sum(xi^2))
```

5.4.5 M2 (C++)

```
int i, n=3, k=(n+1)/2;
double RADIUS=1, A, B, R2, POINT[n], SQ_OF_NORM, P, S, T;
T=pow(drand(),1.0/n);
for(i=1; i<=k-1; i++){
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  SQ_OF_NORM=RADIUS*(1-pow(R2,2.0/(n-2*i)));
  P=sqrt(SQ_OF_NORM/R2);
  POINT[2*i-2]=T*P*A;
  POINT[2*i-1]=T*P*B;
  RADIUS=RADIUS-SQ_OF_NORM;
}
if(n%2){
  S=0.5-drand();
  POINT[n-1]=T*sqrt(RADIUS)*S/fabs(S);
}
else{
  R2=2;
```

```
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  P=sqrt(RADIUS/R2);
  POINT[n-2]=T*P*A;
  POINT[n-1]=T*P*B;
}
```

5.4.6 M3 (C++)

```
int i, n=3, k=k/2;
double A, B, R2, NORMAL[n], P, NN=0;
for(i=1;i<=k;i++){
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  P=sqrt(-2*log(R2)/R2);
  NORMAL[2*i-2]=P*A;
  NORMAL[2*i-1]=P*B;
  NN=NN+P*P*(A*A+B*B);
}
if(n%2){
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  NORMAL[n-1]=sqrt(-2*log(R2)/R2)*A;
  NN=NN+P*P*A*A;
}
NN=NN-2*log(drand());
for(i=1;i<=n;i++)
  NORMAL[i-1]=NORMAL[i-1]/sqrt(NN);
```

5.4.7 M3 (R)

```
n<-3
xi<-rnorm(n+2)
X<-xi[1:n]/sqrt(sum(xi^2))
```

5.4.8 M4 (C++)

```
int i, n=3, k=(n+1)/2;
double RADIUS=1, A, B, R2, POINT[n], SQ_OF_NORM, P, S, T;
for(i=1; i<=k-1; i++){
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  SQ_OF_NORM=RADIUS*(1-pow(R2,2.0/(n+2-2*i)));
  P=sqrt(SQ_OF_NORM/R2);
  POINT[2*i-2]=P*A;
  POINT[2*i-1]=P*B;
  RADIUS=RADIUS-SQ_OF_NORM;
}
if(n%2){
  S=2*drand()-1;
  POINT[n-1]=sqrt(RADIUS)*S;
}
else{
  R2=2;
  while(R2>1){
    A=2*drand()-1;
    B=2*drand()-1;
    R2=A*A+B*B;
  }
  P=sqrt(RADIUS);
  POINT[n-2]=P*A;
  POINT[n-1]=P*B;
}
```

5.5 Tables

n	New	Classic	n	New	Classic
2	0.329	0.785	27	7.004	10.824
3	0.812	1.571	28	7.071	10.838
4	0.854	1.591	29	7.545	11.307
5	1.233	2.148	30	7.519	11.424
6	1.356	2.311	31	7.958	12.143
7	1.786	3.017	32	8.132	12.286
8	1.859	3.127	33	8.500	12.947
9	2.326	3.739	34	8.675	13.041
10	2.488	3.913	35	9.031	13.664
11	2.829	4.482	36	9.147	13.848
12	2.869	4.636	37	9.538	14.308
13	3.264	5.219	38	9.592	14.523
14	3.395	5.315	39	10.027	15.191
15	3.834	6.047	40	10.129	15.449
16	3.933	6.198	41	10.682	15.871
17	4.399	6.747	42	10.990	17.064
18	4.478	6.880	43	11.240	16.941
19	4.810	7.568	44	12.313	19.673
20	4.915	7.747	45	11.642	18.052
21	5.327	8.312	46	11.679	17.607
22	5.468	8.437	47	12.214	18.705
23	5.971	9.192	48	12.294	18.597
24	6.100	9.302	49	13.476	20.506
25	6.423	10.028	50	12.905	19.241
26	6.632	10.009			

Table 5.1: Comparison of the new and the classic algorithm for generating from uniform distribution on the unit n-sphere.

n	**M1**	**M2**	**M3**	**M4**
2	0.998	0.551	0.954	0.271
3	1.654	0.938	1.560	0.702
4	1.770	1.074	1.736	0.808
5	2.416	1.450	2.382	1.213
6	2.544	1.557	2.508	1.332
7	3.171	2.004	3.106	1.698
8	3.301	2.044	3.279	1.800
9	4.013	2.506	3.999	2.336
10	4.137	2.597	4.057	2.329
11	4.738	2.939	4.618	2.721
12	4.854	3.087	4.818	2.809
13	5.489	3.465	5.396	3.245
14	5.612	3.590	5.677	3.414
15	6.254	4.003	6.237	3.781
16	6.403	4.178	6.339	3.863
17	7.101	4.561	6.973	4.278
18	7.157	4.605	7.148	4.444
19	7.860	5.062	7.908	4.847
20	8.148	5.159	7.919	4.929
21	8.725	5.522	8.476	5.299
22	8.878	5.629	8.688	5.376
23	9.215	5.991	9.289	5.725
24	9.554	6.101	9.500	5.836
25	10.170	6.566	9.962	6.306
26	10.329	6.602	10.198	6.356
27	11.967	7.569	12.124	6.977
28	11.459	7.163	11.045	6.829
29	11.967	7.972	12.166	7.473
30	11.821	7.806	11.793	7.485
31	12.380	8.008	12.327	7.741
32	12.595	8.113	12.548	7.839
33	13.113	8.628	13.059	8.330
34	13.357	8.659	13.281	8.415
35	13.983	9.050	13.927	8.838
36	14.001	9.187	14.024	8.826
37	14.751	9.649	14.691	9.350
38	14.848	9.750	14.789	9.417
39	15.387	10.108	15.336	9.697
40	15.514	10.178	15.509	9.816
41	16.207	10.633	16.169	10.231
42	16.360	10.645	16.286	10.399
43	17.130	11.464	16.980	10.689
44	17.242	11.346	17.034	10.794
45	17.822	11.613	17.677	11.331
46	17.936	11.715	17.891	11.449
47	18.550	12.157	18.460	11.887
48	18.551	12.309	18.627	11.889
49	19.394	12.573	19.157	12.344
50	19.431	12.802	19.359	12.553

Table 5.2: Comparison of M1, M2, M3 and M4.

n	M1	M3	n	M1	M3
2	2.153	1.607	27	3.338	2.824
3	2.215	1.685	28	3.417	2.917
4	2.278	1.732	29	3.416	2.933
5	2.324	1.778	30	3.463	3.026
6	2.387	1.872	31	3.495	2.980
7	2.402	1.888	32	3.588	3.042
8	2.418	1.903	33	3.619	3.134
9	2.480	1.966	34	3.635	3.182
10	2.527	1.996	35	3.682	3.214
11	2.606	2.075	36	3.713	3.229
12	2.652	2.074	37	3.790	3.260
13	2.636	2.152	38	3.869	3.307
14	2.668	2.200	39	3.838	3.370
15	2.714	2.293	40	3.869	3.416
16	2.854	2.278	41	3.963	3.463
17	2.933	2.371	42	4.009	3.495
18	2.917	2.434	43	3.993	3.510
19	2.980	2.496	44	4.025	3.541
20	3.010	2.527	45	4.087	3.635
21	3.074	2.558	46	4.134	3.681
22	3.166	2.605	47	4.197	3.713
23	3.198	2.621	48	4.180	3.729
24	3.261	2.761	49	4.212	3.775
25	3.245	2.730	50	4.306	3.837
26	3.307	2.745			

Table 5.3: Comparison of M1 and M3.

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