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Applications of the Atiyah-Singer Index Theorem

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Contents

Introduction

As the title says, this work deals with apllications of the Atiyah-Singer index theorem, especially in the theory of quaternionic manifolds. The theorem relates analytical properties of compact smooth manifolds to their topological properties via the notion of an elliptic differential operator or complex and its analytical index. Although defined purely analytically, this index may be computed from topological data by a formula given by the Atiyah-Singer index theorem and thus provides global information on the manifold. For example, the basic elliptic complex on each smooth manifold is the de Rham complex and its analytical index equals the Euler characteristic. Similarly, the easiest elliptic complex on complex manifolds is the Dolbeault complex whose index is the Todd genus – this result is also known as the Hirzebruch-Riemann-Roch theorem [15]. An analogue of the Dolbeault complex for quaternionic manifolds is the Salamon's complex constructed in [20]. However, it seems that its index has not been yet computed except for some special cases of quaternionic Kähler manifolds [16]. Therefore, the first aim of our work was to compute the analytical index of the Salamon's complex in full generality.

The Salamon's complex is just one of many elliptic complexes arising naturally on quaternionic manifolds. A class of such complexes was described in [6] ending with some computations of their indices in the hyperkähler case. Recently, a much broader class of quaternionic complexes was constructed in [12] in the framework of parabolic geometries and it was shown that many of them are elliptic. Not long after that, a question has been raised whether one can compute their indices and what information on the manifold we could obtain. For example, the index formula provides some integrality conditions on the existence of a quaternionic structure on the manifold. This was our second task.

It has turned out that both these problems may be solved in a uniform way using the Atiyah-Singer index theorem resulting in a procedure for an explicit calculation of the indices. To our knowledge, the results obtained by this procedure are new and has not appeared elsewhere. The aim of this work is first to outline the theory of elliptic complexes on quaternionic manifolds and then describe the procedure for computing their indices.

The structure of the work is as follows. The first short section introduces the notion of an elliptic differential operator and an elliptic complex and the analytical index. The aim of the second section is to outline the theory of quternionic manifolds and the construction of the quaternionic complexes in question. The third section describes a method of Borel and Hirzebruch [8] for computing the Chern classes of complex vector bundles. This is our main technical tool for the calculations. The fourth section presents the Atiyah-Singer index theorem for compact oriented manifolds. In the fifth section we will focus on elliptic complexes which are induced by a G-structure on the manifold. In that case one obtains a simpler version of the Atiyah-Singer theorem to be used in our procedure. At the end of this section we will compute the index of the Salamon's complex for hyperkäler manifolds. The sixth section deals with quaternionic structures from the topological viewpoint [9]. Finally, in the seventh section we will describe the procedure for computing the indices of quaternionic complexes and illustrate it on several simple examples. The results obtained here are completely new.

From the third section onwards we assume some knowledge of characteristic classes for vector bundles (see for example [14]) and the representation theory of compact Lie groups (see fo example [1]). Moreover, the reader should be familiar with the notion of a classifying space of a Lie group.

1. Differential operators and the analytical index

In this first introductory section we will give definitions of differential operators and their symbols and also of the analytical index for elliptic complexes, which is the main object of our interest. It turns out that the index often provides topological information about the manifold itself, a simple motivation for this is given at the end.

Throughout the section we will use the multiindex notation. A multiindex of dimension n is an n-tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of nonnegative integers. The length $|\alpha|$ of α is the sum $\alpha_1 + \ldots + \alpha_n$. The symbol D^α will stand for the partial derivative $\partial^{|\alpha|}/(\partial^{\alpha_1} \ldots \partial^{\alpha_n})$.

To define a differential operator we will start with the notion of a jet. Let $f, g\colon M \to N$ be two smooth maps between smooth manifolds and let $x \in M$. We say that f and g determine the same jet of order $k \in \mathbb{N}_0$ at x if in some, and so any, coordinate chart around x the partial derivatives of f and g coincide up to order k. This is an equivalence relation on the set of maps $f\colon M \to N$, the class of a map f is denoted by $j^k_x f$ and called the *k-jet* of f at x.

We will apply this notion to (local) sections of a complex[1] vector bundle $p\colon E \to M$ over a manifold M. Write $J^k_x(E)$ for the set of k-jets at $x \in M$ of sections $s \in \Gamma E$. $J^k_x(E)$ is a complex vector space under the operations:

$$c \cdot j^k_x s = j^k_x(c \cdot s), \qquad j^k_x s + j^k_x t = j^k_x(s + t).$$

It turns out that the disjoint union $J^k(E) = \bigcup_{x \in M} J^k_x(E)$ with the projection $j^k_x s \mapsto x$ is a complex vector bundle over M. A local trivialization over a coordinate chart $U \subset M$ is given by the partial derivatives of the sections restricted to U, where we can view them as functions from U to the standard fibre of E, i.e. $j^k_x s \mapsto (x, D^\alpha s(x), |\alpha| \le k)$. If $s \in \Gamma E$, then the assignment $x \mapsto j^k_x s$ defines a smooth section of $J^k(E)$. The resulting complex linear operator $j^k\colon \Gamma E \to \Gamma J^k(E)$, $s \mapsto j^k s$, is called the *k-jet prolongation*.

Definition 1.1. Let E^1, E^2 be two complex vector bundles over a smooth manifold M. A complex linear map $D\colon \Gamma E^1 \to \Gamma E^2$ is called a *differential operator* of order k if for all $s \in \Gamma E^1$ and $x \in M$ the equality $j^k_x s = 0$ implies $Ds(x) = 0$.

It follows from the definition that the differential operator D factors uniquely through $J^k(E^1)$. More precisely, for $x \in M$ the mapping $s \mapsto Ds(x)$ is a linear map $\Gamma E^1 \to E^2_x$ and there is a unique linear map $\widetilde{D}_x\colon J^k_x(E^1) \to E^2_x$ such that $Ds(x) = \widetilde{D}_x \circ j^k_x(s)$. Then the induced map $\widetilde{D}\colon J^k(E^1) \to E^2$ is a morphism of vector bundles and $Ds(x) = \widetilde{D}(j^k_x s)$ for all $s \in \Gamma E^1$ and $x \in M$.

Example 1.2. Let $E \to M$ be a vector bundle. The k-jet prolongation $j^k\colon \Gamma E \to \Gamma J^k(E)$ is a differential operator of order k. It is called the *universal operator* of order k for the bundle E. This terminology is justified by the preceding paragraph.

Example 1.3. Let U be an open set in $\mathbb{R}^d$. Then each complex vector bundle over U is of the form $E = U \times \mathbb{C}^m$ and the space ΓE may be identified with the space $C^\infty(U, \mathbb{C}^m)$

[1]We could equally well deal with real vector bundles but because the index theorem is about differential operators between complex vector bundles, we will restrict to these ones.

of smooth complex vector valued functions on U. A *partial differential operator*, in short a PDO, of order k on U is a linear map $D\colon C^\infty(U,\mathbb{C}^m)\to C^\infty(U,\mathbb{C}^n)$ defined by

$$Df(y)=\sum_{|\alpha|\le k}A_\alpha(y)D^\alpha f(y),$$

where $A_\alpha\colon U\to \mathrm{Hom}(\mathbb{C}^m,\mathbb{C}^n)$ are smooth maps. If $f\in C^\infty(U,\mathbb{C}^m)$, then $j^k_y f=0$ is equivalent to vanishing of all the partial derivatives $D^\alpha f(y), |\alpha|\le k$. Hence, a PDO as above is a differential operator of order k.

A differential operator is a *local operator*, i.e. if $s,t\in\Gamma E^1$ are such that $s=t$ on some open set $U\subset M$, then $Ds=Dt$ on U. It is thus natural to ask how a differential operator looks like locally.

Let $\varphi\colon V\to U\subset\mathbb{R}^d$ be a coordinate chart on M such that E^1 and E^2 are both trivial over V. A smooth map $f\colon U\to\mathbb{C}^m$ can be viewed then as a local section of E_1. This can be extended to a global section s and by applying the differential operator D we obtain a linear map $D_U\colon C^\infty(U,\mathbb{C}^m)\to C^\infty(U,\mathbb{C}^n)$. For each $x\in V$ there is a linear map $\widetilde{D}_x\colon J^k_x(E^1)\to E^2_x$ such that $Ds(x)=\widetilde{D}_x\circ j^k_x(s)$. In the coordinate chart V the k-jet $j^k_x s$ corresponds to

$$(D^\alpha f(y))_{|\alpha|\le k}\in\bigoplus_{|\alpha|\le k}\mathbb{C}^m\cong J^k_x(E^1),$$

where $y=\varphi(x)$. Then there are linear mappings $A_\alpha(y)\colon\mathbb{C}^m\to\mathbb{C}^n$ such that $D_U f(y)=\sum_{|\alpha|\le k}A_\alpha(y)D^\alpha f(y)$. One can verify that $A_\alpha(y)$ depend smoothly on y and thus D_U is a PDO of order k.

We have shown that a differential operator of order k is locally a PDO of order k. In fact, the converse holds true: if $D\colon\Gamma E^1\to\Gamma E^2$ is a local operator which locally induces a PDO of order k, then D is a differential operator of order k.

Now we are going to define the symbol of a differential operator. Recall first that a map $\varphi\colon\mathbb{V}_1\to\mathbb{V}_2$ between two vector spaces is called *polynomial of degree* k if in some basis $e_1,e_2,\dots,e_n$ of $\mathbb{V}_1$ it is given by

$$\varphi\left(\sum_{j=1}^n x_je_j\right)=\sum_{|\alpha|=k}x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}\cdot v_\alpha,$$

where $v_\alpha\in\mathbb{V}_2$. The symbol of a differential operator will be a smooth section of the bundle $P^k(T^*M,L(E^1,E^2))$, i.e. for each $x\in M$ we will get a polynomial mapping of degree k on T^*_xM with values in linear maps $E^1_x\to E^2_x$.

Definition 1.4. Let $D\colon\Gamma E^1\to\Gamma E^2$ be a differential operator of order k. The *symbol* σ_D of D is defined as follows. For $v\in T^*_xM$ and $e\in E^1_x$ take $g\in C^\infty(M,\mathbb{R})$ with $g(x)=0$ and $dg_x=v$ and $s\in\Gamma E^1$ with $s(x)=e$. Then

$$\sigma_D(v)e=D\left(\frac{1}{k!}g^k\cdot s\right)(x).$$

One should check that the definition of σ_D does not depend on the choices made and that it is an element of $\Gamma P^k(T^*M,L(E^1,E^2))$. However, this is quite technical and so we refer the reader to [19]. Note that the symbol may also be viewed as a vector bundle morphism between the pullback bundles p^*E^1 and p^*E^2, where $p\colon T^*M \to M$ is the projection. This will be important later.

Example 1.5. To get some insight into what the symbol is, we will compute it locally. Let us have a PDO of order k as in Example 1.3. Put $g(y) = \sum_{j=1}^{d} v_j(y_j - x_j)$ and $f(y) \equiv e$. Then clearly $g(x) = 0$, $dg_x = v$ and

$$\sigma_D(v)e = D\left(\frac{1}{k!}\, g^k \cdot s\right)(x) = \frac{1}{k!}\sum_{|\alpha|\le k} A_\alpha(x)(D^\alpha g^k(x)\cdot e) =$$
$$= \frac{1}{k!}\sum_{|\alpha|=k} A_\alpha(x)(k!\, v^\alpha \cdot e) = \sum_{|\alpha|=k} v^\alpha A_\alpha(x)e.$$

Hence the symbol is given by $\sigma_D(v) = \sum_{|\alpha|=k} v^\alpha A_\alpha(x)$, which is the algebraic counterpart of the leading term of D.

Using the local formulas for differential operators and their symbols it can be shown that they compose well. If $D_1\colon \Gamma E^1 \to \Gamma E^2$ and $D_2\colon \Gamma E^2 \to \Gamma E^3$ are differential operators of orders k and l, then $D_2 \circ D_1$ is a differential operator of order $k+l$ and for the symbols we have

$$\sigma_{D_2\circ D_1}(v) = \sigma_{D_2}(v)\circ\sigma_{D_1}(v)$$

for all $v \in T^*M$.

A differential operator $D\colon \Gamma E^1 \to \Gamma E^2$ is called *elliptic* if for all $x \in M$ and $v \in T^*_xM$, $v \neq 0$, the linear map $\sigma_D(v)\colon E^1_x \to E^2_x$ is an isomorphism.

A slightly more general notion is that of an *elliptic complex*. This is a finite sequence of differential operators

$$D : 0 \to \Gamma E^0 \xrightarrow{D_0} \Gamma E^1 \xrightarrow{D_1} \Gamma E^2 \xrightarrow{D_2} \cdots \xrightarrow{D_{r-1}} \Gamma E^r \to 0$$

such that $D_j \circ D_{j-1} = 0$ for all $j = 1, 2, \ldots, r$ and, moreover, we require that the sequence of symbols

$$0 \to E^0_x \xrightarrow{\sigma_{D_0}(v)} E^1_x \xrightarrow{\sigma_{D_1}(v)} E^2_x \xrightarrow{\sigma_{D_2}(v)} \cdots \xrightarrow{\sigma_{D_{r-1}}(v)} E^r_x \to 0$$

is exact for all $x \in M$ and $v \in T^*_xM$, $v \neq 0$.

A nice and important property of elliptic complexes is that on compact manifolds the cohomology groups $H^j(D) = \operatorname{Ker} D_j/\operatorname{Im} D_{j-1}$ are finite dimensional, for the proof see [3]. We define the *analytical index* of an elliptic complex D as its Euler characteristic, i.e.

$$\text{a-ind}\, D = \sum_{j=0}^{r} (-1)^j \dim_{\mathbb{C}} H^j(D).$$

In the case of a single elliptic operator $D\colon \Gamma E^1 \to \Gamma E^2$, which is just an elliptic complex of length one, the definition reduces to

$$\text{a-ind}\, D = \dim_{\mathbb{C}} \operatorname{Ker} D - \dim_{\mathbb{C}} \operatorname{Coker} D.$$

Elliptic complexes are quite common in differential geometry, the basic examples are the de Rham complex or the Dolbeault complex. We are, however, mainly interested in elliptic complexes which arise quite naturally on quaternionic manifolds. These will be studied in the next section.

Example 1.6. Let M be a compact manifold and let $\Omega^j = \Gamma(\Lambda^j T^*M \otimes \mathbb{C})$ be the space of complex-valued j-forms on M. It follows from the local description of the exterior derivative d that it is a first order differential operator and then the de Rham complex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m \to 0$$

is a complex of differential operators.

Let us compute the symbol of d. Take a smooth function g with $g(x) = 0$ and $dg_x = v$ and a j-form $\phi \in \Omega^j$ with $\phi(x) = e$. Then

$$\sigma_d(v)e = d(g \cdot \phi)(x) = dg_x \wedge \phi(x) + g(x) \cdot d\phi(x) = v \wedge e$$

i.e. $\sigma_d(v)\colon \Lambda^j T^*_x M \otimes \mathbb{C} \to \Lambda^{j+1} T^*_x M \otimes \mathbb{C}$ is just the exterior product on v.

Lemma 1.7. *Let V be an n-dimensional real or complex vector space and let $v \in V$. Consider the following complex of linear maps*

$$0 \to \Lambda^0 V \xrightarrow{v\wedge -} \Lambda^1 V \xrightarrow{v\wedge -} \Lambda^2 V \xrightarrow{v\wedge -} \cdots \xrightarrow{v\wedge -} \Lambda^n V \to 0.$$

This complex is exact if and only if $v \neq 0$.

Proof. If $v = 0$, then the complex is clearly not exact. Let $v \neq 0$ and assume that $(v, v_2, \ldots, v_n)$ is a basis of V. An arbitrary $e \in \Lambda^j V$ can be written in the form

$$e = \sum_{1<i_2<\ldots<i_j\leq n} e_{1 i_2 \ldots i_j} v \wedge v_{i_2} \wedge \ldots \wedge v_{i_j} + \sum_{1<i_1<\ldots<i_j\leq n} e_{i_1 i_2 \ldots i_j} v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_j}.$$

Then we obtain

$$v \wedge e = \sum_{1<i_1<\ldots<i_j\leq n} e_{i_1 i_2 \ldots i_j} v \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_j}.$$

Therefore, if $v \wedge e = 0$, then $e_{i_1 i_2 \ldots i_j} = 0$ for $i_1 > 1$ and

$$e = v \wedge \Big(\sum_{1<i_2<\ldots<i_j\leq n} e_{1 i_2 \ldots i_j} v_{i_2} \wedge \ldots \wedge v_{i_j} \Big).$$

In particular, $e \in \operatorname{Im}(v \wedge -)$ and the complex is exact. □

The lemma now implies that the de Rham complex is an elliptic complex. Its analytical index is well-known – the cohomology groups are just the de Rham cohomology groups $H^*_{de\,R}(M; \mathbb{C})$ and the index is the usual Euler characteristic $\chi(M)$.

This example shows that the analytical index of an elliptic complex, although defined purely from analytical data, may carry some topological information about the manifold. This is the meaning of the Atiyah-Singer index theorem, which gives a topological formula for the analytical index.

2. Quaternionic manifolds and quaternionic complexes

The aim of this section is to introduce quaternionic manifolds and to show that there is a large class of elliptic complexes over them. These complexes arise as subcomplexes of the so-called curved BGG-sequences, which are constructed using the theory of parabolic geometries. We will outline the basic notions and constructions and then state the results for quaternionic manifolds.

We will start with the classical definition of quaternionic manifolds via G-structures. Consider $\mathbb{R}^{4m}$ as the space $\mathbb{H}^m$ of m-tuples of quaternions. Then the group $\mathrm{Sp}(1)$ of unit quaternions acts on it by $a \cdot v = v\bar{a}$ and $\mathrm{GL}(m, \mathbb{H})$ acts by left matrix multiplication. These actions induce injections $\mathrm{Sp}(1) \hookrightarrow \mathrm{GL}(4m, \mathbb{R})$ and $\mathrm{GL}(m, \mathbb{H}) \hookrightarrow \mathrm{GL}(4m, \mathbb{R})$. Now we define the group $\mathrm{Sp}(1)\mathrm{GL}(m, \mathbb{H})$ as the product of these groups in $\mathrm{GL}(4m, \mathbb{R})$. More abstractly, $\mathrm{Sp}(1)\mathrm{GL}(m, \mathbb{H})$ is isomorphic to the quotient $\mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{GL}(m, \mathbb{H})$, that is the pair $(A, a) \in \mathrm{Sp}(1) \times \mathrm{GL}(m, \mathbb{H})$ represents the mapping $v \mapsto Av\bar{a}$. In the sequel we will write simply G_0 instead of $\mathrm{Sp}(1)\mathrm{GL}(m, \mathbb{H})$.

Definition 2.1. A $4m$-dimensional manifold M, $m \geq 2$, is called *almost quaternionic* if it has a G_0-structure $\mathcal{P}$, i.e. there is a principal G_0-bundle $\mathcal{P}$ and an isomorphism of vector bundles $\mathcal{P} \times_{G_0} \mathbb{R}^{4m} \cong TM$. An almost quaternionic manifold is called *quaternionic* if the G_0-structure $\mathcal{P}$ admits a torsion-free connection.

Let G_1 denote the group $\mathrm{Sp}(1) \times \mathrm{GL}(m, \mathbb{H})$, which is the double cover of G_0. The bundle $\mathcal{P}$ can be locally lifted to a principal G_1-bundle $\mathcal{P}_1$. If $\rho \colon G_1 \to \mathrm{Aut}(\mathbb{V})$ is a representation of G_1, we can construct the associated vector bundle $V = \mathcal{P}_1 \times_{G_1} \mathbb{V}$. This vector bundle exists globally if either the lift can be done globally, or the action ρ factors through G_0.

Recall that the standard complex $\mathrm{GL}(m, \mathbb{H})$-module (or $\mathrm{Sp}(m)$-module) is defined as follows. View $\mathbb{H}^m$ as a right vector space over $\mathbb{H}$ with the scalar multiplication being the usual multiplication from right. Then, restricting scalars to $\mathbb{C}$, left multiplication by matrices in $\mathrm{GL}(m, \mathbb{H})$ is a complex linear map and so a complex representation.

Let $\mathbb{E}$ and $\mathbb{F}$ denote the standard complex $\mathrm{Sp}(1)$-module and $\mathrm{GL}(m, \mathbb{H})$-module, respectively, and define a map $\varphi \colon \mathbb{R}^{4m} = \mathbb{H}^m \to \mathbb{E} \otimes_{\mathbb{C}} \mathbb{F}$ by $\varphi(u) = \mathrm{j} \otimes u - 1 \otimes u\mathrm{j}$. This is a real linear map which maps the real basis $e_l, e_l\mathrm{i}, e_l\mathrm{j}, e_l\mathrm{k}, 1 \leq l \leq m$, of $\mathbb{H}^m$ to a complex basis of $\mathbb{E} \otimes_{\mathbb{C}} \mathbb{F}$. Moreover, by a direct computation one can verify that φ is G_0-equivariant. It follows that the complexification of the G_0-module $\mathbb{H}^m$ is isomorphic to $\mathbb{E} \otimes_{\mathbb{C}} \mathbb{F}$ and then we have the following isomorphisms of complex vector bundles

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} \cong \mathcal{P} \times_{G_0} (\mathbb{E} \otimes_{\mathbb{C}} \mathbb{F}) \cong \mathcal{P}_1 \times_{G_1} (\mathbb{E} \otimes_{\mathbb{C}} \mathbb{F}) \cong E \otimes_{\mathbb{C}} F. \tag{2.2}$$

However, note that the bundles E and F may not exist globally.

The above decomposition gives rise to a natural subcomplex of the de Rham complex, which we are going to describe now. Since irreducible complex $\mathrm{Sp}(1)$-modules are precisely the symmetric powers $S^j\mathbb{E}$, there exist irreducible $\mathrm{GL}(m, \mathbb{H})$-modules $\mathbb{L}^j_k$ such that the G_1-module $\Lambda^j(\mathbb{E} \otimes \mathbb{F})$ decomposes as

$$\Lambda^j(\mathbb{E} \otimes \mathbb{F}) \cong \bigoplus_{k=0}^{[j/2]} S^{j-2k}\mathbb{E} \otimes \mathbb{L}^j_k.$$

It turns out that $\mathbb{L}_0^j$ is the exterior power $\Lambda^j\mathbb{F}$. Therefore, for all $0 \le j \le 2n$, the G_1-module $\Lambda^j(\mathbb{E}\otimes\mathbb{F})$ contains a G_1-submodule

$$\mathbb{A}^j \cong S^j\mathbb{E}\otimes\Lambda^j\mathbb{F}.$$

This is in fact a G_0-module because the action of G_1 on $S^j\mathbb{E}\otimes\Lambda^j\mathbb{F}$ factors through G_0. But this together with the isomorphism from (2.2) implies that there is a natural vector subbundle $A^j \cong \mathcal{P}\times_{G_0}\mathbb{A}^j$ in $\Lambda^j(T^*M\otimes\mathbb{C})$. Let d denote the exterior derivative on complex-valued differential forms and $p_j\colon \Lambda^j(T^*M\otimes\mathbb{C})\to A^j$ the projection. If we put $D_j = p_j\circ d$, we obtain the following sequence of differential operators

$$0\to\Gamma A^0 \xrightarrow{D_1} \Gamma A^1 \xrightarrow{D_2} \Gamma A^2 \xrightarrow{D_3} \cdots \xrightarrow{D_{2m}} \Gamma A^{2m}\to 0. \tag{2.3}$$

This sequence is closely related to the (almost) quaternionic structure of the manifold.

Theorem 2.4 (Salamon)**.** *An almost quaternionic manifold M is quaternionic if and only if the sequence (2.3) is a complex. If this is the case, then the complex is elliptic.*

Proof. The proof can be found in [20], Theorem 4.1. □

This statement is very similar to the corresponding statement about almost complex and complex manifolds and the Dolbeault complex. For the quaternionic manifolds, the elliptic complex (2.3) will be called the *Salamon's complex*. It is the simplest case of a much wider class of elliptic complexes constructed as subcomplexes of the BGG-sequences. We will describe the BGG-sequences in a general setting of parabolic geometries and then return to the quaternionic case to obtain the complexes in question. However, we will not give many details – these can be found in [12] and [11], the general theory then in [10] – and so we encourage the reader who is more interested in the method of computing indices to skip the rest of the section and return later.

Parabolic geometries form a subclass of the more general Cartan geometries with some additional structure on the Lie algebra $\mathfrak{g}$. So let us start with the Cartan geometries. We will need some notation first. If $\mathcal{G}$ is a principal H-bundle, then we denote by $r^h\colon\mathcal{G}\to\mathcal{G}$ the principal right action of $h\in H$ on $\mathcal{G}$ and by $\zeta_X(u) = \frac{d}{dt}|_{t=0}u\cdot\exp(tX)$ the fundamental vector field on $\mathcal{G}$ determined by $X\in\mathfrak{h}$.

Definition 2.5. Let $H\subset G$ be a Lie subgroup of a Lie group G and let $\mathfrak{g}$ be the Lie algebra of G. A *Cartan geometry* of type (G,H) on a manifold M is a principal H-bundle $p\colon\mathcal{G}\to M$ together with a $\mathfrak{g}$-valued one-form $\omega\in\Omega^1(\mathcal{G},\mathfrak{g})$, called the *Cartan connection*, which satisfies

(1) $(r^h)^*\omega = \mathrm{Ad}(h^{-1})\circ\omega$ for all $h\in H$,
(2) $\omega(\zeta_X(u)) = X$ for all $X\in\mathfrak{h}$,
(3) $\omega|_{T_u\mathcal{G}}\colon T_u\mathcal{G}\to\mathfrak{g}$ is a linear isomorphism for all $u\in\mathcal{G}$.

The first two properties say that ω is H-equivariant and that it reproduces the fundamental vector fields.

The simplest example of a Cartan geometry of type (G,H) is the *homogeneous model*, which is the natural bundle $p\colon G\to G/H$ endowed with the left Maurer-Cartan form $\omega\in\Omega^1(\mathcal{G},\mathfrak{g})$ as the Cartan connection.

The *curvature form* of a given Cartan geometry $(\mathcal{G} \to M, \omega)$ is a $\mathfrak{g}$-valued two-form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)], \quad \xi, \eta \in T_u\mathcal{G}.$$

Note that the Maurer-Cartan equation implies that the curvature of the homogeneous model vanishes identically. The form K is equivariant and horizontal, hence it may be viewed as a two-form $\kappa \in \Omega^2(M, \mathcal{A}M)$ on M with values in the *adjoint tractor bundle* $\mathcal{A}M = \mathcal{G} \times_H \mathfrak{g}$, here the action of H on $\mathfrak{g}$ is given by restricting the adjoint action of G.

Exactly as for the homogeneous model, the Cartan connection induces an isomorphism $TM \cong \mathcal{G} \times_H \mathfrak{g}/\mathfrak{h}$, the action of H on $\mathfrak{g}/\mathfrak{h}$ is again induced by the adjoint action. The projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ then induces a projection $\Pi\colon \mathcal{A}M \to TM$ and the TM-valued two-form $\kappa_- = \Pi \circ \kappa$ on M is called the *torsion* of the Cartan geometry. The geometry is called *torsion-free* if this torsion vanishes.

Now we can move to the notion of a graded Lie algebra and then proceed to the definition of parabolic geometries.

Definition 2.6. A $|k|$-*grading* on a semisimple Lie algebra $\mathfrak{g}$ is a direct sum decomposition into linear subspaces

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and such that the Lie subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ is generated as a Lie algebra by $\mathfrak{g}_{-1}$.

The subspaces $\mathfrak{g}^i = \mathfrak{g}_i \oplus \ldots \oplus \mathfrak{g}_k$ give a filtration of $\mathfrak{g}$ and $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$. In particular, $\mathfrak{p} = \mathfrak{g}^0$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_+ = \mathfrak{g}^1$ is a nilpotent ideal in $\mathfrak{p}$.

If G is a Lie group with the Lie algebra $\mathfrak{g}$, we define its subgroups $G_0 \subset P \subset G$ by

$$\begin{aligned} G_0 &= \{g \in G \,|\, \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for all } -k \leq i \leq k\}, \\ P &= \{g \in G \,|\, \mathrm{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \text{ for all } -k \leq i \leq k\}. \end{aligned}$$

One can verify that G_0 and P have Lie algebras $\mathfrak{g}_0$ and $\mathfrak{p}$, respectively. The group P is called the *parabolic subgroup* of G and G_0 the *Levi subgroup* of P. Moreover, let P_+ be the image of $\mathfrak{p}_+$ in the exponential map $\exp\colon \mathfrak{p} \to P$. Then P_+ is a normal nilpotent subgroup of P and it may be shown that $P/P_+ \cong G_0$. In particular, P is the semidirect product of the subgroups G_0 and P_+.

Definition 2.7. A *parabolic geometry* on a manifold M is a Cartan geometry on M of type (G, P), where P is the parabolic subgroup in a semisimple Lie group G.

Many classical geometric structures on manifolds such as conformal, quaternionic or CR-structures, may be described as a parabolic geometry of certain type.

Example 2.8. Put $\mathfrak{g} = \mathfrak{sl}(m+1, \mathbb{H})$, the Lie algebra of quaternionic matrices with the real trace equal zero. Define a $|1|$-grading on $\mathfrak{g}$ as follows. We have

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z \\ X & A \end{pmatrix} \,\middle|\, a \in \mathbb{H},\, X, Z^T \in \mathbb{H}^n,\, A \in \mathrm{Mat}_m(\mathbb{H}),\, \mathrm{Re}\, a + \mathrm{Re}(\mathrm{tr}(A)) = 0 \right\}$$

and the gradation looks like

$$\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \in \mathfrak{g}_{-1}, \quad \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{g}_0, \quad \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_1 .$$

As the Lie group G take $\mathrm{PGL}(m+1, \mathbb{H})$, the quotient of all invertible quaternionic linear endomorphisms of $\mathbb{H}^{m+1}$ by the closed normal subgroup of all real multiples of the identity. Then the parabolic subgroup P is the image in the quotient of the stabilizer of the quaternionic line in $\mathbb{H}^{m+1}$ spanned by the first basis vector and $G_0 \cong \mathrm{Sp}(1)\mathrm{GL}(m, \mathbb{H})$. The homogeneous model G/P can be identified with the *quaternionic projective space* $\mathbb{H}\mathrm{P}^m$.

Under some regularity and normality conditions the parabolic geometry on M is determined by an underlying geometric structure on M. In the case of $|1|$-gradings this is nothing but a G_0-structure on M in the usual sense. Indeed, since P acts freely on $\mathcal{G}$, the same is true for the subgroup $P_+ \subset P$ and we can consider the orbit space $\mathcal{G}_0 = \mathcal{G}/P_+$. Then $\mathcal{G}_0 \to M$ is a principal bundle with the structure group $P/P_+ \cong G_0$. Recall that the tangent bundle of M may be written as $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, where the action of P on $\mathfrak{g}/\mathfrak{p}$ is induced by the adjoint action on $\mathfrak{g}$. But for $|1|$-graded geometries the group P_+ acts trivially on $\mathfrak{g}/\mathfrak{p}$ and so we also get $TM \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}/\mathfrak{p}$, i.e. M admits a G_0-structure.

In particular, a parabolic geometry of the type as in Example 2.8 above induces an almost quaternionic structure on the manifold. On the other hand, it can be shown that an almost quaternionic manifold admits a unique regular normal parabolic geometry of that type and, moreover, this geometry is torsion-free if and only if the manifold is quaternionic. For details and proofs see [10].

Having the basic notions in hand, we can proceed to outline the construction of the BGG-sequences. Let $(p\colon \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and let $\mathbb{W}$ be a real or complex G-module. Then $\mathbb{W}$ is also a P-module and infinitesimally it is a representation of the Lie algebra $\mathfrak{p}_+$. Consider a mapping $\partial^*\colon \Lambda^{k+1}\mathfrak{p}_+ \otimes \mathbb{W} \to \Lambda^k\mathfrak{p}_+ \otimes \mathbb{W}$ defined on decomposamble elements by

$$\begin{aligned}\partial^*(Z_0 \wedge \ldots \wedge Z_k \otimes v) = &\sum_{i=1}^{k} (-1)^{i+1} Z_0 \wedge \ldots \wedge \widehat{Z}_i \wedge \ldots \wedge Z_k \otimes Z_i \cdot v + \\ &+ \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_0 \wedge \ldots \wedge \widehat{Z}_i \wedge \ldots \wedge \widehat{Z}_j \wedge \ldots \wedge Z_k \otimes v,\end{aligned}$$

where the hats denote omissions and the dot $\cdot$ is the infinitesimal action of $\mathfrak{p}_+$ on $\mathbb{W}$. One can verify that $\partial^* \circ \partial^* = 0$ and so we get a complex, the differential ∂^* is called the *Kostant codifferential*. Because all spaces in the complex are P-modules and ∂^* is P-equivariant, the homology groups $H_k(\mathfrak{p}_+, \mathbb{W})$ are naturally P-modules as well. In fact, the subgroup P_+ acts trivially on the homology and so the representation comes from the subgroup G_0.

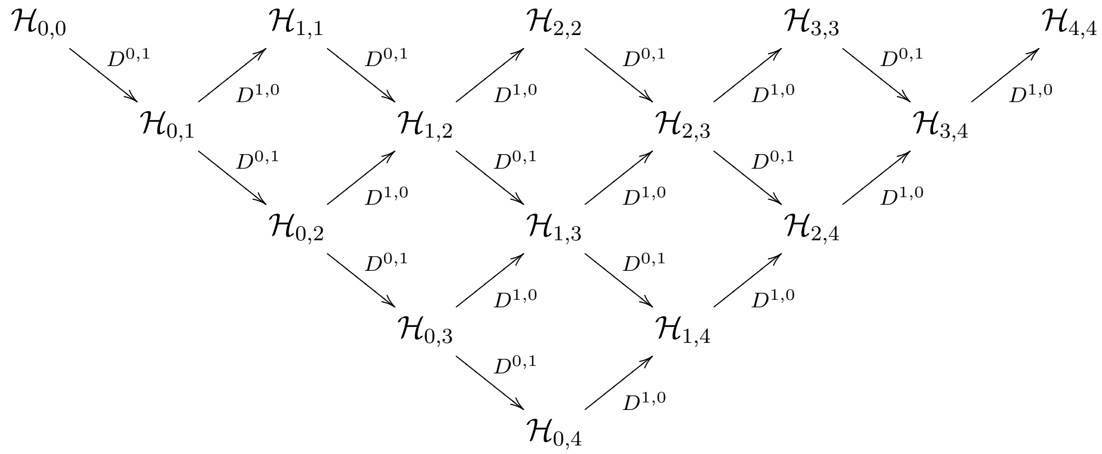
The Killing form of the Lie algebra $\mathfrak{g}$ induces an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of P-modules. Hence $T^*M \cong \mathcal{G} \times_P \mathfrak{p}_+$ and $\Lambda^k T^*M \otimes W \cong \mathcal{G} \times_P (\Lambda^k\mathfrak{p}_+ \otimes \mathbb{W})$ and the P-equivariancy of ∂^* implies that we obtain a vector bundle map $\partial^*\colon \Lambda^{k+1}T^*M \otimes W \to \Lambda^k T^*M \otimes W$. Again $\partial^* \circ \partial^* = 0$ and the kernel and the image of ∂^* are subbundles. Their quotient is by construction isomorphic to the associated vector bundle $\mathcal{G}_0 \times_{G_0} H_k(\mathfrak{p}_+, \mathbb{W})$, which we denote by $H_k(T^*M, W)$. These bundles appear in the *curved Bernstein-Gelfand-Gelfand* *sequence* (in short the BGG-sequence)

$$\cdots \to \Gamma(H_{k-1}(T^*M, W)) \xrightarrow{D^{\mathbb{W}}} \Gamma(H_k(T^*M, W)) \xrightarrow{D^{\mathbb{W}}} \Gamma(H_{k+1}(T^*M, W)) \to \cdots .$$

The differential operators $D^{\mathbb{W}}$ are constructed in [11]. An important property of these operators is that they are *strongly invariant* (see [11]), which in particular implies that their symbol is induced from P-equivariant polynomial maps $\mathfrak{p}_+ \to L(H_k(\mathfrak{p}_+, \mathbb{W}), H_{k+1}(\mathfrak{p}_+, \mathbb{W}))$.

The compostition $D^{\mathbb{W}} \circ D^{\mathbb{W}}$ is nontrivial in general and so the BGG-sequence is not a complex. However, by splitting the G_0-modules $H_k(\mathfrak{p}_+, \mathbb{W})$ into a direct sum of irreducible components we obtain the corresponding splitting of $H_k(T^*M, W)$ into a direct some of subbundles and then the BGG-operators break into components acting between the irreducible pieces. For torsion-free geometries one can find simple algebraic conditions under which the composition of these components vanishes and hence gives rise to a subcomplex in the BGG-sequence. This is the main result of [12].

The irreducible components of $H_k(\mathfrak{p}_+, \mathbb{W})$ may be described via the so-called Hasse graph of the parabolic subalgebra $\mathfrak{p}$. In the case of quaternionic structures (see Example 2.8) this graph has a triangular shape as is shown below for $m = 2$. Here $H_k(\mathfrak{p}_+, \mathbb{W}) = \oplus_{i+j=k} \mathcal{H}_{i,j}$ is the decomposition into irreducible components and the arrows denote the splitting of the BGG-operator $D^{\mathbb{W}}$ into a sum $D^{1,0} + D^{0,1}$ of two operators $D^{1,0}\colon \Gamma\mathcal{H}_{i,j} \to \Gamma\mathcal{H}_{i+1,j}$ and $D^{0,1}\colon \Gamma\mathcal{H}_{i,j} \to \Gamma\mathcal{H}_{i,j+1}$.



The subcomplexes we are looking for are precisely the compositions of the $D^{0,1}$-operators and the $D^{1,0}$-operators.

Theorem 2.9 (Čap, Souček)**.** *Let M be a quaternionic manifold of dimension $4m$ and $(\mathcal{G} \to M, \omega)$ the corresponding torsion-free regular normal parabolic geometry of type $(\mathrm{PGL}(m+1, \mathbb{H}), P)$. Then for each irreducible $\mathrm{PGL}(m+1, \mathbb{H})$-module $\mathbb{W}$ the BGG-sequence associated to the vector bundle $W = \mathcal{G} \times_P \mathbb{W}$ contains the following subcomplexes*

$$\Gamma\mathcal{H}_{j,j} \xrightarrow{D^{0,1}} \Gamma\mathcal{H}_{j,j+1} \xrightarrow{D^{0,1}} \dots \xrightarrow{D^{0,1}} \Gamma\mathcal{H}_{j,2m}, \quad \text{for } j = 0, 1, \dots, 2m-2,$$

$$\Gamma\mathcal{H}_{0,j} \xrightarrow{D^{1,0}} \Gamma\mathcal{H}_{1,j} \xrightarrow{D^{1,0}} \cdots \xrightarrow{D^{1,0}} \Gamma\mathcal{H}_{j,j}, \quad \text{for } j = 2, 3, \cdots, 2m.$$

Proof. The proof can be found in [12]. □

We have thus obtained a class of quaternionic complexes for each choice of the irreducible $\mathrm{PGL}(m+1,\mathbb{H})$-module $\mathbb{W}$. However, only some of them are known to be elliptic. Let $\mathbb{V}$ be the standard complex $\mathrm{GL}(m+1,\mathbb{H})$-module and put $\mathbb{W}_k = S^k\mathbb{V}^* \otimes S^k\mathbb{V}$. Then $\mathbb{W}_k$ is a $\mathrm{PGL}(m+1,\mathbb{H})$-module, because the action by the real multiples of the identity is trivial. We will now describe more closely the $D^{0,1}$-subcomplex of the corresponding BGG-sequence starting at $\mathcal{H}_{0,0}$, i.e. the left edge of the triangle above. Denote by $\mathbb{W}_k^j$ the representation of the group $G_0 = \mathrm{Sp}(1)\mathrm{GL}(m,\mathbb{H})$ inducing the vector bundle $\mathcal{H}_{0,j}$. Furthermore, let $\mathbb{E}$ and $\mathbb{F}$ be the standard complex $\mathrm{Sp}(1)$- and $\mathrm{GL}(m,\mathbb{H})$-modules, respectively. According to [20] the character ring of complex representations of the group $\mathrm{GL}(m,\mathbb{H})$ is isomorphic to that of the group $\mathrm{U}(2m) \subset \mathrm{GL}(2m,\mathbb{C})$. In particular, we have a theory of highest weights for $\mathrm{GL}(m,\mathbb{H})$. Now one can show (see [12]) that

$$\mathbb{W}_k^j = S^{j+k}\mathbb{E} \otimes (\Lambda^j\mathbb{F} \otimes S^k\mathbb{F}^*)_0 \quad \text{for } j < 2m, \quad \mathbb{W}_k^{2m} = S^{2(m+k)}\mathbb{E} \otimes \Lambda^{2m}\mathbb{F}, \tag{2.10}$$

where the zero subscript denotes the irreducible component of the tensor product with the highest weight being the sum of the highest weights of the respective factors.

Theorem 2.11 (Čap, Souček)**.** *Let M be a $4m$-dimensional quaternionic manifold with a G_0-structure $\mathcal{P}$. Denote by W_k^j the associated vector bundle $\mathcal{P} \times_{G_0} \mathbb{W}_k^j$. Then for each $k \geq 0$ the subcomplex*

$$0 \to \Gamma W_k^0 \xrightarrow{D^{0,1}} \Gamma W_k^1 \xrightarrow{D^{0,1}} \cdots \xrightarrow{D^{0,1}} \Gamma W_k^{2m} \to 0$$

of the BGG-sequence is elliptic.

Proof. For the proof see again [12]. □

In fact, one can prove that also the right edge of the triangle, i.e. the $D^{1,0}$-subcomplex starting at $\mathcal{H}_{0,2m}$, is elliptic. However, the left and right edges are dual to each other and so the analytical indices are equal.

Note that by setting $k = 0$ we obtain precisely the Salamon's complex (2.3). The goal of the rest of this thesis is to develop a technique for computing the analytical indices of all the elliptic complexes from Theorem 2.11.

3. Characteristic classes and representations

This section is devoted to the theory of characteristic classes of principal G-bundles and associated vector bundles. We will follow the approach of Borel and Hirzebruch ([8]), which relies on the representation theory of compact Lie groups. This will be technically very useful later, when we have to compute the Chern classes and the Chern character of some complex vector bundles associated to a principal G-bundle.

Let G be a compact Lie group. Recall that a *maximal torus* T in G is a maximal connected abelian subgroup of G and that the *Weyl group* of G is the group of automorphisms of T which are restrictions of inner automorphisms of G. Let $\mathfrak{t}$ be the Lie algebra of T. Then the exponential map $\exp\colon \mathfrak{t} \to T$ is the universal covering of T. The inverse image of the identity element will be denoted $\Lambda_T = \exp^{-1}(e)$ and called the *unit lattice* of T. This is a free commutative group of rank $\dim T$. A real valued linear form on $\mathfrak{t}$ is called an *integral weight* if it takes integral values on Λ_T.

The unit lattice Λ_T may be identified with the fundamental group $\pi_1(T)$ as follows. Choose $u_0 \in \mathfrak{t}$ and for each $u \in \Lambda_T$ take a path $\gamma_u\colon [0,1] \to \mathfrak{t}$ joining u_0 with $u_0 + u$. Then $\exp \circ \gamma_u$ is a loop in T and its homotopy class does not depend on the choice of γ_u. One can see from lifting properties of covering spaces and the fact that $\mathfrak{t}$ is simply connected that the resulting map is an isomorphism.

Because $\pi_1(T)$ is abelian as the fundamental group of a Lie group, it is isomorphic to $H_1(T;\mathbb{Z})$ and so is the unit lattice. But then $H^1(T;\mathbb{Z}) \cong \mathrm{Hom}(H_1(T;\mathbb{Z}),\mathbb{Z})$ is isomorphic to the group of integral weights.

Furthermore, there is an isomorphism $H^1(T;\mathbb{Z}) \to H^2(BT;\mathbb{Z})$ which commutes with the action of the Weyl group of G. This can be explicitly described as minus the transgression map in the universal principal T-bundle (see for example [18]). Then it follows that $H^{**}(BT;\mathbb{Z}) \cong \mathbb{Z}[[t_1, t_2, \dots, t_n]]$, the ring of formal power series. Here $t_j \in H^2(BT;\mathbb{Z})$ are the images of a basis of integral weights and $H^{**}(X)$ stands for the product $\prod_{j=0}^{\infty} H^j(X)$ of the cohomology groups.

The rational cohomology ring of the classifying space BG can be described as a subring in $H^{**}(BT;\mathbb{Q})$. However, we will restrict our attention to the classifying space of the unitary group $\mathrm{U}(n)$, which enables us to use the integer coefficients.

Proposition 3.1 (Borel)**.** *Let T be a maximal torus of the unitary group $\mathrm{U}(n)$. Then the inclusion $\iota\colon T \hookrightarrow \mathrm{U}(n)$ induces an injection $(B\iota)^{**}\colon H^{**}(B\mathrm{U}(n);\mathbb{Z}) \to H^{**}(BT;\mathbb{Z})$ whose image is the subring of elements invariant under the action of the Weyl group of $\mathrm{U}(n)$.*

Proof. The proof can be found in [7] or [18]. □

As a maximal torus T of $\mathrm{U}(n)$ we may take the set of all diagonal matrices of the form

$$\begin{pmatrix} \exp(2\pi \mathrm{i} x_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \exp(2\pi \mathrm{i} x_n) \end{pmatrix}.$$

Then $x_1, x_2, \dots, x_n$ viewed as forms on the Lie algebra $\mathfrak{t}$ of T constitute a basis of integral weights. The Weyl group of $U(n)$ is the symmetry group S_n, it permutes the weights $x_1, x_2, \dots, x_n$ and therefore also the elements $t_1, t_2, \dots, t_n \in H^{**}(BT;\mathbb{Z})$. The above theorem implies that $H^{**}(B\mathrm{U}(n);\mathbb{Z}) \cong \mathbb{Z}[[c_1, c_2, \dots, c_n]]$ where c_j is the j-th elementary symmetric polynomial in $t_1, t_2, \dots, t_n$.

Now we are ready to give a definition of Chern classes of complex vector bundles. Recall that n-dimensional complex vector bundles are in bijective correspondence with principal $\mathrm{U}(n)$-bundles and so we can define the Chern classes for the principal bundles.

Let $p\colon \mathcal{P} \to X$ be a principal $\mathrm{U}(n)$-bundle and let $T \subset \mathrm{U}(n)$ be the maximal torus as above. If $E = \mathcal{P} \times_{\mathrm{U}(n)} \mathbb{C}^n$ is the vector bundle associated to $\mathcal{P}$, then the quotient bundle $\pi\colon \mathcal{P}/T \to X$ is precisely the bundle of flags in fibres of E. Here by a flag we mean an ordered n-tuple of mutually orthogonal lines in $\mathbb{C}^n$. The classical splitting principle for complex vector bundles (see [14]) says that the pullback vector bundle π^*E decomposes into a sum of line bundles and, moreover, the induced map $\pi^*\colon H^*(X;\mathbb{Z}) \to H^*(\mathcal{P}/T;\mathbb{Z})$ is injective. The space $\mathcal{P}$ is also the base space of the principal T-bundle $\mathcal{P} \to \mathcal{P}/T$. Let $x_1, x_2, \dots, x_n$ be a basis of integral weights as before and put $y_j = -\tau(x_j) \in H^2(\mathcal{P}/T;\mathbb{Z})$, where $\tau\colon H^1(T;\mathbb{Z}) \to H^2(\mathcal{P}/T;\mathbb{Z})$ is the transgression map in the bundle $\mathcal{P} \to \mathcal{P}/T$. We define the *total Chern class* $c(\mathcal{P})$ of $\mathcal{P}$ by

$$\pi^*(c(\mathcal{P})) = \prod_{j=1}^{n}(1+y_j).$$

As noted before, the map π^* is injective and so the element $c(\mathcal{P})$ is uniquely determined once we know that the right-hand side lies in the image of π^*. For this, let $f\colon X \to B\mathrm{U}(n)$ be the classifying map for $\mathcal{P}$. Then we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{P} & \longrightarrow & \mathcal{P}/T & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow & & \downarrow f \\ E\mathrm{U}(n) & \longrightarrow & E\mathrm{U}(n)/T & \xrightarrow{\pi'} & B\mathrm{U}(n). \end{array}$$

But $E\mathrm{U}(n) \to E\mathrm{U}(n)/T$ is a principal T-bundle and $E\mathrm{U}(n)$ is contractible. Therefore $E\mathrm{U}(n)/T$ is precisely BT and π' is $B\iota$, where $\iota\colon T \hookrightarrow \mathrm{U}(n)$ is the inclusion. Now the claim follows from the previous description of $H^{**}(B\mathrm{U}(n);\mathbb{Z})$.

Writing $c(\mathcal{P}) = 1 + c_1(\mathcal{P}) + \ldots + c_n(\mathcal{P})$ in its homogeneous components we also get the individual *Chern classes* $c_j(\mathcal{P}) \in H^{2j}(X;\mathbb{Z})$. These are mapped by π^* to the elementary symmetric polynomials in y_j. Furthermore, we define the *Chern character* $\mathrm{ch}(\mathcal{P})$ and the *Todd class* $\mathrm{td}(\mathcal{P})$ by

$$\pi^{**}(\mathrm{ch}(\mathcal{P})) = \sum_{j=1}^{n} e^{y_j} = \sum_{j=0}^{\infty}\frac{1}{j!}\big(y_1^j + \ldots + y_n^j\big), \quad \pi^{**}(\mathrm{td}(\mathcal{P})) = \prod_{j=1}^{n}\frac{y_j}{1-e^{-y_j}}.$$

Because the expressions on the right-hand side are symmetric in the y_j's, one can write both $\mathrm{ch}(\mathcal{P})$ and $\mathrm{td}(\mathcal{P})$ as power series in the Chern classes $c_1(\mathcal{P}), c_2(\mathcal{P}), \ldots, c_n(\mathcal{P})$, for more details see Appendix A.

Note that all these characteristic classes are natural with respect to principal bundle maps, i.e. if $f\colon (\mathcal{P}_1 \to X_1) \to (\mathcal{P}_2 \to X_2)$ is a principal bundle map then $\underline{f}^* c(\mathcal{P}_2) = c(\mathcal{P}_1)$ and similarly for the other two. This follows from the fact that principal bundle maps commute with transgressions.

By considering associated vector bundles to principal $\mathrm{U}(n)$-bundles we may view the Chern classes, the Chern character and the Todd class as defined on complex vector bundles. It can be verified (see [8]) that these definitions are equivalent to the more standard ones. The usefulness of this particular approach will be apparent from the following.

Let G be a compact Lie group and $S \subset G$ its maximal torus. Let $\lambda\colon G \to \mathrm{U}(n)$ be a complex representation of G such that $\lambda(S) \subset T$, where $T \subset \mathrm{U}(n)$ is the maximal torus as before. Then we have the induced map $\lambda^*\colon H^1(T;\mathbb{Z}) \to H^1(S;\mathbb{Z})$. Let $x_1, x_2, \ldots, x_n$ be the basis of $H^1(T;\mathbb{Z})$. The elements $\omega_j = \lambda^*(x_j)$, viewed either as elements of $H^1(S;\mathbb{Z})$ or integral weights, are called the *weights* of λ. Of course, for each $s \in S$ the matrix $\lambda(s)$ is diagonal with entries $\exp(2\pi \mathrm{i}\omega_j)$, i.e. ω_j are the weights of the representation λ in its usual sense. If $\mathcal{P}$ is a principal G-bundle, we may construct its λ-*extension* $\mathcal{P}_\lambda = \mathcal{P} \times_\lambda \mathrm{U}(n)$, which is a principal $\mathrm{U}(n)$-bundle. Here the action of G on $\mathrm{U}(n)$ is given by $g \cdot a = \lambda(g)a$. The following proposition will be crucial for all our computations.

Proposition 3.2 (Borel, Hirzebruch)**.** *Let $\lambda\colon G \to \mathrm{U}(n)$ be a complex representation of a compact Lie group G such that $\lambda(S) \subset T$, where S is a maximal torus in G. Let ω_j be the weights of λ. For a principal G-bundle $\mathcal{P} \to X$ consider its λ-extension $\mathcal{P}_\lambda$. Denote by $\eta\colon \mathcal{P}/S \to X$ the projection and put $w_j = -\tau(\omega_j)$, where $\tau\colon H^1(S;\mathbb{Z}) \to H^2(\mathcal{P}/S;\mathbb{Z})$ is the transgression map in the principal S-bundle $\mathcal{P} \to \mathcal{P}/S$. Then*

$$\eta^*(c(\mathcal{P}_\lambda)) = \prod_{j=1}^{n}(1 + w_j),$$

$$\eta^{**}(\mathrm{ch}(\mathcal{P}_\lambda)) = \sum_{j=1}^{n} e^{w_j}, \quad \eta^{**}(\mathrm{td}(\mathcal{P}_\lambda)) = \prod_{j=1}^{n} \frac{w_j}{1 - e^{-w_j}}.$$

Proof. Consider the λ-map $\phi\colon \mathcal{P} \to \mathcal{P}_\lambda = \mathcal{P} \times_\lambda \mathrm{U}(n)$ defined by $\phi(p \cdot g^{-1}) = [p \cdot g^{-1}, \lambda(g)]$. Then we have the following commutative diagram

$$\begin{array}{ccccccc} S & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{P}/S & \xrightarrow{\eta} & X \\ {\scriptstyle\lambda}\downarrow & & {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\mathrm{id}_X} \\ T & \longrightarrow & \mathcal{P}_\lambda & \longrightarrow & \mathcal{P}_\lambda/T & \xrightarrow{\pi} & X. \end{array}$$

By the definition of Chern classes we have

$$\pi^*(c(\mathcal{P}_\lambda)) = \prod_{j=1}^{n}(1 - \tau(x_j)),$$

where τ is the transgression map in the principal T-bundle $\mathcal{P}_\lambda \to \mathcal{P}_\lambda/T$. Because the map ϕ_1^* commutes with transgressions we obtain

$$\eta^*(c(\mathcal{P}_\lambda)) = \phi_1^* \circ \pi^*(c(\mathcal{P}_\lambda)) = \prod_{j=1}^{n}(1 - \phi_1^* \circ \tau(x_j)) = \prod_{j=1}^{n}(1 - \tau \circ \lambda^*(x_j)) = \prod_{j=1}^{n}(1 + w_j)$$

and this is the first claimed formula. The other two may be proved similarly. □

The maps η^* and η^{**} may not be injective in general. However, there are two special cases in which both these maps are injective. The first is if $\mathcal{P}$ is a principal $\mathrm{U}(m)$-bundle, as we already know by the splitting principle. The second is if $\mathcal{P}$ is the universal principal G-bundle $EG \to BG$ for some classical Lie group G and we consider the rational cohomology groups. Indeed, from EG we get a principal S-bundle $EG \to EG/S$ with the total space EG being contractible, hence EG/S is the classifying space BS and then $\eta\colon BS = EG/S \to BG$ is the classifying map $B\iota$ induced by the inclusion $\iota\colon S \hookrightarrow G$. It is shown in [7] that for the classical groups $G = \mathrm{SO}(m)$ or $G = \mathrm{Sp}(m)$ the induced map $(B\iota)^{**}$ is injective at least over the rationals. In these cases we may formally ignore η^* or η^{**} and write the formulas in Proposition 3.2 as formal equalities in the cohomology ring of the base space BG.

By considering associated bundles to the principal bundles we obtain the same result for complex vector bundles. As an easy illustration of the proposition we will now compute the Chern classes of the exterior powers of the standard representation of $\mathrm{U}(m)$.

Example 3.3. Let $x_1, x_2, \dots, x_m$ be the standard basis of integral weights of $\mathrm{U}(m)$ and let $0 \le k \le m$ be an integer. Consider the standard representation $\lambda\colon \mathrm{U}(m) \to \mathrm{U}(\binom{m}{k})$ of $\mathrm{U}(m)$ on the k-th exterior power $\Lambda^k\mathbb{C}^m$. If (e_j) is the canonical basis of $\mathbb{C}^m$, then the products $e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}$ constitute a basis of $\Lambda^k\mathbb{C}^m$ and the action of an element $x \in S$ in the maximal torus is given by

$$\lambda(x)(e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}) = \exp[(2\pi\mathrm{i})(x_{j_1} + x_{j_2} + \dots + x_{j_k})] \cdot e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}.$$

Hence the weights of λ are the sums $x_{j_1} + x_{j_2} + \dots + x_{j_k}$, where $1 \le j_1 < j_2 < \dots < j_k \le m$.

Now if $\mathcal{P} \to X$ is a principal $\mathrm{U}(m)$-bundle, then $\mathcal{P}_\lambda$ is the principal frame bundle of the k-th exterior power of the complex vector bundle associated to $\mathcal{P}$. The Chern classes of these bundles are given by

$$c(\mathcal{P}) = \prod_{j=1}^{m}(1 + y_j), \quad c(\mathcal{P}_\lambda) = \prod_{1 \le j_1 < \dots < j_k \le m} (1 + y_{j_1} + \dots + y_{j_k}).$$

The first formula follows from the definition and the second by applying Proposition 3.2. We also ignore the map η^* as was said before. Then the Chern classes $c_l(\mathcal{P})$ are precisely the elementary symmetric polynomials in y_j's and the Chern classes $c_l(\mathcal{P}_\lambda)$ are some symmetric polynomials in y_j's and thus they can be written as polynomials in $c_l(\mathcal{P})$'s.

If $E = \mathcal{P} \times_{\mathrm{U}(m)} \mathbb{C}^m$ is the complex vector bundle associated to $\mathcal{P}$, then $\mathcal{P}_\lambda \times_{\mathrm{U}(\binom{m}{k})} \mathbb{C}^{\binom{m}{k}}$ is the exterior power $\Lambda^k E$. For further purposes we will need the following simple formula for the Chern character of the formal polynomial $\Lambda_t(E) = \sum_{k=0}^{m} t^k \Lambda^k E \in H^{**}(X;\mathbb{Q})$

$$\mathrm{ch}(\Lambda_t(E)) = \sum_{k=0}^{m} t^k \left(\sum_{1 \le j_1 < \dots < j_k \le m} e^{y_{j_1} + \dots + y_{j_k}} \right) = \prod_{j=1}^{m}(1 + te^{y_j}). \tag{3.4}$$

Proposition 3.2 can also be used to prove that for two complex vector bundles E_1 and E_2 we have the relations

$$\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2), \quad \mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1)\,\mathrm{ch}(E_2),$$
$$\mathrm{td}(E_1 \oplus E_2) = \mathrm{td}(E_1)\,\mathrm{td}(E_2).$$

Indeed, if $\mathbb{V}_1$ and $\mathbb{V}_2$ are the corresponding G-modules, then the weights of $\mathbb{V}_1 \oplus \mathbb{V}_2$ are precisely the weights of $\mathbb{V}_1$ and $\mathbb{V}_2$ altogether and the weights of $\mathbb{V}_1 \otimes \mathbb{V}_2$ are the sums of the weights of $\mathbb{V}_1$ with the weights of $\mathbb{V}_2$.

4. K-theory and the topological index

In this section we will first outline the relation between symbols of elliptic complexes and the K-theory and then state the Atiyah-Singer index theorem, which will be our basic tool for computing the analytical indices. However, it is not the aim to give a proof of the index theorem.

Let X be a compact topological space. Isomorphism classes of complex vector bundles over X form an abelian semigroup under the Whitney sum and we define the *K-group* $K(X)$ of X as the corresponding Grothendieck group. More explicitly, $K(X)$ is the quotient of the free group generated by isomorphism classes $[E]$ of complex vector bundles under the equivalence relation $[E^1 \oplus E^2] \sim [E^1] + [E^2]$. Then $K(X)$ is a commutative ring under the tensor product of vector bundles. A continuous map $f\colon X \to Y$ between two compact spaces induces a ring homomorphism $f^*\colon K(Y) \to K(X)$ by taking pullback bundles.

If X has a given basepoint x_0, we define the *reduced K-group* $\widetilde{K}(X)$ as the kernel of the homomorphism $i^*\colon K(X) \to K(x_0)$ induced by the inclusion $i\colon x_0 \hookrightarrow X$. Then $\widetilde{K}(X)$ is a ring generally without an identity.

For a locally compact space X we define $K(X) = \widetilde{K}(X^+)$ where X^+ is the one-point compactification of X. If X is actually compact, then $X^+ = X \sqcup x_0$ and $\widetilde{K}(X \sqcup x_0) \cong K(X)$, i.e. the two definitions coincide.

There is an equivalent definition of $K(X)$ for all locally compact spaces X which will be suitable for our purposes. By a *complex E^* of vector bundles* on X we mean a collection of vector bundle morphisms $\alpha_j\colon E^j \to E^{j+1}$, $0 \le j \le r$, called *differentials*, such that for each $x \in X$ the sequence

$$0 \to E^0_x \xrightarrow{\alpha_0} E^1_x \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{r-1}} E^r_x \to 0 \tag{4.1}$$

is a complex, i.e. $\alpha_j \circ \alpha_{j-1} = 0$. A *morphism* $f\colon E^* \to F^*$ of two complexes $E^* = (E^j, \alpha_j)$ and $F^* = (F^j, \beta_j)$ is a collection of vector bundle morphisms $f_j\colon E^j \to F^j$ such that $f_j \circ \alpha_j = \beta_j \circ f_{j-1}$. The *support* of a complex E^* is the set of points $x \in X$ for which the sequence (4.1) is not exact. A complex E^* is called *acyclic* if it has empty support, i.e. (4.1) is exact for all x.

Denote by $L(X)$ the set of isomorphism classes of complexes on X with compact support. Then $L(X)$ is an abelian semigroup under the direct sum. Two elements E^*_0 and E^*_1 of $L(X)$ are called *homotopic* if there is $E^* \in L(X \times [0,1])$ such that $E^*_0 \cong E^*|_{X\times\{0\}}$ and $E^*_1 \cong E^*|_{X\times\{1\}}$. We say that $E^*_0, E^*_1 \in L(X)$ are *equivalent* and write $E^*_0 \sim E^*_0$ if there are acyclic complexes $F^*_0, F^*_1 \in L(X)$ such that $E^*_0 \oplus F^*_0$ and $E^*_1 \oplus F^*_1$ are homotopic. It is easy to verify that $\sim$ is an equivalence relation and we have the following important result.

Proposition 4.2 (Atiyah, Segal)**.** *Let X be a locally compact space. Then $L(X)/\sim$ is an abelian group naturally isomorphic to $K(X)$.*

Proof. A very nice proof using some homological algebra can be found in [23]. □

If X is actually compact, then of course each complex has compact support and the isomorphism is simply given by $E^* \mapsto \sum_{j=0}^{r} (-1)^j E^j$. This map is certainly surjective and injectivity follows from the fact that each complex on a compact space is homotopic to a complex of length one with the zero differential.

Example 4.3. Let M be a compact manifold and D an elliptic complex over it. Then the symbol σ_D of D is a complex of vector bundles on T^*M and its support is precisely the zero section of the projection $p\colon T^*M \to M$, which is isomorphic to M, hence compact. Therefore σ_D defines a class in $L(T^*M)$ and thus also in $K(T^*M) \cong K(TM)$ – here T^*M is identified with TM by means of a Riemannian metric.

It can be shown that the analytical index of an elliptic complex D depends only on its symbol class σ_D in $K(TM)$. In other words, if the symbols of two elliptic complexes on M determine the same class in $K(TM)$, then their analytical indices are equal. It follows that the analytical index defines correctly a mapping from some subset of $K(TM)$ to $\mathbb{Z}$.

If we consider a broader class of operators called *pseudodifferential operators*, then this mapping is defined on the whole of $K(TM)$ and, moreover, it is a homomorphism of rings. Let us denote it by a-ind: $K(TM) \to \mathbb{Z}$ and call the *analytical index*.

The idea of the Atiyah-Singer index theorem is to study this homomorphism more abstractly. For each compact manifold X there is a naturally defined ring homomorphism t-ind: $K(TM) \to \mathbb{Z}$, called the *topological index*, and this can be uniquely characterized by several axioms. The proof of the index theorem then goes along to verify that these axioms are also satisfied by the analytical index a-ind and so this must coincide with the topological index t-ind. All the details can be found in the article [4].

Although defined in K-theory, the topological index can be translated into cohomology using the Chern character. Because the Chern character preserves Whitney sums and tensor products and depends only on the isomorphism class of the vector bundle, it can be extended to a ring homomorphism $\mathrm{ch}\colon K(X) \to H^*(X;\mathbb{Q})$ for X compact and $\mathrm{ch}\colon \widetilde{K}(X^+) \to \widetilde{H}^*(X^+;\mathbb{Q})$ for X locally compact. For an arbitrary compact manifold M the one-point compactification of its tangent bundle TM is homeomorphic to the quotient bundle BM/SM, where BM and SM are the unit ball and unit sphere bundles of TM, respectively. Denote by $\psi\colon H^*(M;\mathbb{Q}) \to H^*(BM, SM;\mathbb{Q}) \cong \widetilde{H}^*(TM^+;\mathbb{Q})$ the Thom isomorphism and by $[M]$ the fundamental class of M. The Atiyah-Singer index theorem may now be stated as follows.

Theorem 4.4 (Atiyah, Singer)**.** *Let D be an elliptic complex on a compact oriented manifold M of dimension m and let $\sigma_D \in K(TM)$ be its symbol class. Then*

$$\text{a-ind}\, D = \text{t-ind}\, \sigma_D = (-1)^{m(m+1)/2}\{\psi^{-1}(\mathrm{ch}\, \sigma_D) \cdot \mathrm{td}(TM \otimes \mathbb{C})\}[M].$$

Proof. A proof of the first equality is in [4], while the second one is proved in [5]. □

Note that while the left-hand side of the formula above is an integer, the right-hand side is a priory only a rational number. This often provides some integrality conditions on the characteristic classes involved.

From now on we do not need to distinguish between the analytical and the topological index and so we will write only $\mathrm{ind}\, D$ for the analytical index of an elliptic complex D. Although the Atiyah-Singer formula looks rather simple, there is still a problem with computing the Chern character of the symbol class σ_D. We will see in the next section that for some special classes of elliptic complexes related to a G-structure on the manifold this problem can be solved quite easily.

Before we move to this, let us note that there is an immediate corollary of the index theorem, which may be quite surprising.

Corollary 4.5 (Atiyah, Singer)**.** *Let D be an elliptic complex on a compact oriented odd-dimensional manifold M. Then its analytical index is zero.*

Proof. Let $F\colon T^*M \to T^*M$ be the antipodal map, i.e. $F(v) = -v$, and $p\colon T^*M \to M$ the projection. Consider first an elliptic differential operator $d\colon \Gamma E^1 \to \Gamma E^2$ on M of order k. Then its symbol satisfies $\sigma_d(F(v)) = (-1)^k \sigma_d(v)$. However, σ_d and $(-1)^k\sigma_d$ as elements of $L(T^*M)$ are homotopic through

$$T^*M \times [0,1] \ni (v,t) \mapsto e^{\mathrm{i}\pi k t}\sigma_d(v)\colon (p^*E^1)_v \to (p^*E^2)_v.$$

It follows that in $K(T^*M) = K(TM)$ we have $F^*(\sigma_d) = \sigma_d$. Analogously, this holds for an elliptic complex D and its symbol σ_D. In particular, $F^*(\operatorname{ch}\sigma_D) = \operatorname{ch} F^*(\sigma_D) = \operatorname{ch}\sigma_D$. In the cohomology of T^*M, the induced map F^* sends the Thom class τ of T^*M to $(-1)^m\tau$, where $m = \dim M$. Note finally that the base map $\underline{F}$ of F is the identity on M. Assuming that m is odd, from all these observations and naturality we obtain

$$\begin{aligned}\operatorname{ind} D &= (-1)^{m(m+1)/2}\{\psi^{-1}(\operatorname{ch}\sigma_D)\cdot \operatorname{td}(TM\otimes\mathbb{C})\}[M] = \\ &= (-1)^{m(m+1)/2}\underline{F}^*\{\psi^{-1}(\operatorname{ch}\sigma_D)\cdot \operatorname{td}(TM\otimes\mathbb{C})\}\underline{F}_*[M] = \\ &= (-1)^{m(m+1)/2}\{-\psi^{-1}(F^*(\operatorname{ch}\sigma_D))\cdot \operatorname{td}(TM\otimes\mathbb{C})\}[M] = \\ &= -(-1)^{m(m+1)/2}\{\psi^{-1}(\operatorname{ch}\sigma_D)\cdot \operatorname{td}(TM\otimes\mathbb{C})\}[M] = -\operatorname{ind} D.\end{aligned}$$

Because $\operatorname{ind} D$ is an integer, this implies that it must be zero. □

As we saw earlier, the Euler characteristic is the index of the de Rham complex. The corollary is thus a deep generalization of the well-known fact that the Euler characteristic of an odd-dimensional manifold is zero (see [13]).

5. Symbols associated to G-structures

The aim of this section is to develop a technique for computing indices for elliptic complexes whose symbol is in some way induced from a G-structure on the manifold. The idea is to reduce the computation to the classifying space BG of the group G and then apply the theory of characteristic classes as was presented earlier. We will end this section with two simple examples – the de Rham complex and the Salamon's complex on hyperkähler manifolds.

Let M be a compact oriented manifold, G a compact Lie group and $\mathbb{V}$ a real oriented G-module. Suppose that M admits a G-structure, i.e. there is a principal G-bundle $\mathcal{P}$ and an isomorphism of oriented vector bundles $TM \cong \mathcal{P}\times_G \mathbb{V}$. Let $\mathbb{E}^j$, $0\le j\le r$, be complex G-modules and put $E^j = \mathcal{P}\times_G \mathbb{E}^j$. Denote by D an elliptic complex

$$0 \to \Gamma E^0 \xrightarrow{D_0} \Gamma E^1 \xrightarrow{D_1} \cdots \xrightarrow{D_{r-1}} \Gamma E^r \to 0$$

of differential operators on M. Assume further that $\varphi_j\colon \mathbb{V}^* \to L(\mathbb{E}^j, \mathbb{E}^{j+1})$ is a polynomial G-equivariant map such that for all $v\in\mathbb{V}^*, v\neq 0$, the sequence

$$0 \to \mathbb{E}^0 \xrightarrow{\varphi_0(v)} \mathbb{E}^1 \xrightarrow{\varphi_1(v)} \cdots \xrightarrow{\varphi_{r-1}(v)} \mathbb{E}^r \to 0 \tag{5.1}$$

is exact. If the symbol sequence σ_D of D is induced via the isomorphisms $T^*M \cong \mathcal{P}\times_G \mathbb{V}^*$ and $E^j \cong \mathcal{P}\times_G \mathbb{E}^j$ from (5.1), then we say that σ_D is *associated to the G-structure $\mathcal{P}$*. To be more explicit, the symbol is a sequence of fibrewise polynomial bundle maps

$$T^*M = \mathcal{P}\times_G \mathbb{V}^* \to L(E^j, E^{j+1}) = \mathcal{P}\times_G L(\mathbb{E}^j, \mathbb{E}^{j+1})$$

and we require that $[p, v] \mapsto [p, \varphi_j(v)]$.

Example 5.2. Let $G = \mathrm{SO}(m)$ and put $\mathbb{E}^j = \Lambda^j(\mathbb{C}^m)^*$. Then a G-structure on M is induced by a Riemannian metric and an orientation of TM and $E^j = \Lambda^j(T^*M\otimes\mathbb{C})$. Consider the de Rham complex of complex-valued differential forms

$$0 \to \Gamma E^0 \xrightarrow{d} \Gamma E^1 \xrightarrow{d} \Gamma E^2 \xrightarrow{d} \cdots \xrightarrow{d} \Gamma E^m \to 0.$$

As we saw in Section 1, its symbol at $(x, v) \in T^*M$ is given by the exterior product

$$0 \to E_x^0 \xrightarrow{v\wedge -} E_x^1 \xrightarrow{v\wedge -} E_x^2 \xrightarrow{v\wedge -} \cdots \xrightarrow{v\wedge -} E_x^m \to 0.$$

It is now obvious that this symbol is associated to the $\mathrm{SO}(m)$-structure of M via the mappings $\varphi_j(v) = v \wedge -$.

We will need further a general notion of characteristic classes for principal G-bundles.

Definition 5.3. Let G be a compact Lie group. A *characteristic class of principal G-bundles* is a function α assigning to every principal G-bundle $p\colon \mathcal{P} \to X$ a cohomology class $\alpha(\mathcal{P})$ in $H^{**}(X; R) = \prod_{j=0}^{\infty} H^j(X; R)$, where R is some fixed coefficient ring. Moreover, we require that this function is natural with respect to principal bundle maps, i.e. if $f\colon (\mathcal{P}_1 \to X_1) \to (\mathcal{P}_2 \to X_2)$ is a principal bundle map then $\underline{f}^{**}c(\mathcal{P}_2) = c(\mathcal{P}_1)$.

The naturality property implies that there is a bijective correspondence between characteristic classes of principal G-bundles and elements of $H^{**}(BG; R)$, where BG is the classifying space of the group G. Indeed, for each principal G-bundle $\mathcal{P} \to X$ there is an up to homotopy unique map $f\colon X \to BG$ such that $\mathcal{P} \cong f^*(EG)$, the pullback of the universal principal G-bundle. But then $\alpha(\mathcal{P}) = f^{**}\alpha(EG)$ and so α is uniquely determined by $\alpha(EG)$.

We are interested in characteristic classes with rational coefficients. Therefore, an important result for us is that $H^{**}(BG; \mathbb{Q})$ is a ring of formal power series in several indeterminates with rational coefficients, hence an integral domain. This is shown in [7].

Now we can state and prove a simplification of the index theorem, which is appropriate for our purposes.

Proposition 5.4 (Atiyah, Singer). *Let M be a compact oriented manifold of dimension $2m$, G a compact Lie group and $\rho\colon G \to \mathrm{SO}(2m)$ a homomorphism. Assume that M has a G-structure $\mathcal{P}$, i.e. TM is associated to $\mathcal{P}$ via ρ. Let $\mathbb{E}^j$, $0 \le j \le r$, be complex G-modules and let E^j be the corresponding associated vector bundles. Suppose that*

$$0 \to \Gamma E^0 \xrightarrow{D_0} \Gamma E^1 \xrightarrow{D_1} \cdots \xrightarrow{D_{r-1}} \Gamma E^r \to 0$$

is an elliptic complex with its symbol associated to the G-structure $\mathcal{P}$. Let $f\colon M \to BG$ be the classifying map for the bundle $\mathcal{P}$. Put $\widetilde{E}^j = EG \times_G \mathbb{E}^j$ and $\widetilde{V} = EG \times_\rho \mathbb{R}^{2m}$. If the Euler class $e(\widetilde{V})$ is nonzero, then it divides $\sum(-1)^j \mathrm{ch}\, \widetilde{E}_j \in H^{**}(BG; \mathbb{Q})$ and the index of the above complex is given by

$$(-1)^m \left\{ f^{**}\left(\frac{\sum_{j=0}^{r}(-1)^j \mathrm{ch}\, \widetilde{E}^j}{e(\widetilde{V})} \right) \cdot \mathrm{td}(TM \otimes \mathbb{C}) \right\}[M].$$

Proof. We will follow the proof given in [5]. We would like to simplify the term $\psi^{-1}(\mathrm{ch}\, \sigma_D)$ in Theorem 4.4. Denote by $\mathbb{V}$ the real oriented G-module corresponding to the representation ρ. For an arbitrary compact principal G-bundle $\mathcal{P} \to X$ we may construct a complex E^* of vector bundles[2] over $V^* = \mathcal{P} \times_G \mathbb{V}^*$ precisely as in the beginning of the section. Of course, we do not require now that this should be the symbol of an elliptic complex. Then

[2] Throughout the proof, E^* will always stand for a complex of vector bundles as in the preceding section, while V^* will be the dual vector bundle to V.

E^* determines a class in $K(V^*)$ and so we may define $\alpha(\mathcal{P}) = \psi^{-1}(\operatorname{ch} E^*) \in H^*(X;\mathbb{Q})$, where ψ is the Thom isomorphism for the oriented vector bundle $p\colon V^* \to X$. Clearly, $P \mapsto \alpha(\mathcal{P})$ is a characteristic class of principal G-bundles. To prove the proposition we have to compute this class.

The Euler class $e(\widetilde{V}^*) = e(\widetilde{V}) \in H^*(BG;\mathbb{Q})$ determines a characteristic class e of principal G-bundles. Explicitly, $e(\mathcal{P}) = e(\mathcal{P} \times_G \mathbb{V})$. If $i\colon X \to V^*$ is the inclusion as the zero section, then $e(V^*) = i^*\psi(1)$. Therefore, by applying i^* on the Thom isomorphism $\psi(\alpha(\mathcal{P})) = p^*(\alpha(\mathcal{P}))\cdot\psi(1)$ we obtain $i^*(\operatorname{ch} E^*) = \alpha(\mathcal{P})\cdot e(\mathcal{P})$. This implies that the left-hand side is again a characteristic class. But $i^*(\operatorname{ch} E^*) = \operatorname{ch}(i^*E^*)$ and i^*E^* is the following complex of vector bundles on X

$$0 \to \mathcal{P} \times_G \mathbb{E}^0 \xrightarrow{0} \mathcal{P} \times_G \mathbb{E}^1 \xrightarrow{0} \cdots \xrightarrow{0} \mathcal{P} \times_G \mathbb{E}^r \to 0.$$

Because X is compact, this complex determines in $K(X)$ the class $\sum_{j=0}^r (-1)^j \mathcal{P} \times_G \mathbb{E}^j$. In particular, $\operatorname{ch}(i^*E^*) = \sum_{j=0}^r (-1)^j \operatorname{ch}(\mathcal{P} \times_G \mathbb{E}^j)$. It follows that this characteristic class of principal G-bundles is defined by the element $\sum_{j=0}^r (-1)^j \operatorname{ch} \widetilde{E}^j \in H^{**}(BG;\mathbb{Q})$ and from above we have the equality

$$\sum_{j=0}^r (-1)^j \operatorname{ch} \widetilde{E}^j = \alpha(EG)\cdot e(\widetilde{V}).$$

Because $e(\widetilde{V}) \neq 0$ and $H^{**}(BG;\mathbb{Q})$ is an integral domain, we may divide by $e(\widetilde{V})$ to obtain $\alpha(EG)$ and so the charecteristic class α.

Finally, return to the case when $\mathcal{P}$ is the G-structure on the manifold M and E^* is the symbol σ_D. If $\mathcal{P} \cong f^*EG$, then

$$\psi^{-1}(\operatorname{ch}\sigma_D) = \alpha(\mathcal{P}) = \alpha(f^*EG) = f^{**}\alpha(EG) = f^{**}\left(\frac{\sum_{j=0}^r (-1)^j \operatorname{ch} \widetilde{E}^j}{e(\widetilde{V})}\right)$$

and inserting into Theorem 4.4 we get the claimed formula. □

Note that the proposition shows that if we have two elliptic complexes between the same bundles whose symbols are associated to the G-structure on M, then their indices are equal. In particular, the index does not depend on the actual differential operators.

We have seen before that the symbol of the de Rham complex is associated to a $\mathrm{SO}(m)$-structure on M. Similarly, by introducing a Riemannian metric on a quaternionic manifold M, we may endow it with a $\mathrm{Sp}(1)\mathrm{Sp}(m)$-structure and then the symbol of the Salamon's complex (2.3) is associated to the given $\mathrm{Sp}(1)\mathrm{Sp}(m)$-structure. More generally, this holds for all the quaternionic complexes from Theorem 2.11. Therefore, once we show that the Euler class of the universal vector bundle over $B\mathrm{Sp}(1)\mathrm{Sp}(m)$ is nonzero, we can apply the above proposition. We will study quaternionic structures from the topological viewpoint in more detail in the next section.

We will illustrate now the use of Proposition 5.4 together with Proposition 3.2 on two rather simple examples.

Example 5.5. Let M be a compact $2m$-dimensional oriented Riemannian manifold. We will compute the (topological) index of the de Rham complex by applying Proposition 5.4

with $\rho\colon \mathrm{SO}(2m) \to \mathrm{SO}(2m)$ being the identity and $\mathbb{E}^j = \Lambda^j(\mathbb{C}^{2m})^*$. As we know from Section 1, the index must equal the Euler characteristic of the manifold and so we will compare the two results in the end.

Put $\widetilde{V} = E\mathrm{SO}(2m) \times_{\mathrm{SO}(2m)} \mathbb{R}^{2m}$, the universal real vector bundle. The Euler class of this vector bundle is nonzero (see [14]), hence the condition is satisfied. Moreover, we have $\widetilde{E}^j = \Lambda^j(\widetilde{V} \otimes \mathbb{C})^* \cong \Lambda^j(\widetilde{V} \otimes \mathbb{C})$ and so we only need to compute the Chern character of the polynomial $\Lambda_{-1}(\widetilde{V} \otimes \mathbb{C})$. In view of the formula (3.4) it suffices to find the Chern classes of $\widetilde{V} \otimes \mathbb{C}$. We will apply Proposition 3.2.

As a maximal torus S of $\mathrm{SO}(2m)$ we may take the subgroup of block diagonal matrices with blocks $D_1, D_2, \ldots, D_m$ of the form

$$D_j = \begin{pmatrix} \cos 2\pi x_j & -\sin 2\pi x_j \\ \sin 2\pi x_j & \cos 2\pi x_j \end{pmatrix}.$$

Then $x_1, x_2, \ldots, x_m$ viewed as linear forms on the Lie algebra $\mathfrak{s}$ form a basis of integral weights. Consider a representation $\lambda\colon \mathrm{SO}(2m) \to \mathrm{U}(2m)$ given as a composition $\delta \circ \gamma$ of the inclusion $\gamma\colon \mathrm{SO}(2m) \hookrightarrow \mathrm{U}(2m)$ and the inner automorphism $\delta\colon A \mapsto BAB^{-1}$, where B is the block diagonal matrix with blocks

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}.$$

The maximal torus S is mapped onto the diagonal matrices with entries $\exp(\pm 2\pi \mathrm{i} x_j)$, i.e. the forms $\pm x_j$ are the weights of the representation λ. The λ-extension $E\mathrm{SO}(2m)_\lambda$ is the principal frame bundle of the complex vector bundle $\widetilde{V} \otimes \mathbb{C}$. Then we have [3]

$$c\,(\widetilde{V} \otimes \mathbb{C}) = \prod_{j=1}^{m} (1 + y_j)(1 - y_j), \quad \mathrm{ch}(\Lambda_{-1}(\widetilde{V} \otimes \mathbb{C})) = \prod_{j=1}^{m} (1 - e^{y_j})(1 - e^{-y_j})$$

and similarly for the Todd class

$$\mathrm{td}(\widetilde{V} \otimes \mathbb{C}) = \prod_{j=1}^{m} \frac{y_j(-y_j)}{(1 - e^{-y_j})(1 - e^{y_j})}.$$

Now let $f\colon M \to B\mathrm{SO}(2m)$ be the classifying map of the $\mathrm{SO}(2m)$-structure of M. Then $TM \otimes \mathbb{C} \cong f^*(\widetilde{V} \otimes \mathbb{C})$ and so $\mathrm{td}(TM \otimes \mathbb{C}) = f^{**}\,\mathrm{td}(\widetilde{V} \otimes \mathbb{C})$. The topological index of the de Rham complex is given by

$$\begin{aligned} \mathrm{ind} &= (-1)^m f^{**} \left(\frac{\mathrm{ch}(\lambda_{-1}(\widetilde{V} \otimes \mathbb{C}))}{e(\widetilde{V})} \cdot \mathrm{td}(\widetilde{V} \otimes \mathbb{C}) \right) [M] = \\ &= (-1)^m f^{**} \left(\frac{\prod_{j=1}^{m} (1 - e^{y_j})(1 - e^{-y_j})}{e(\widetilde{V})} \cdot \prod_{j=1}^{m} \frac{y_j(-y_j)}{(1 - e^{-y_j})(1 - e^{y_j})} \right) [M] = \\ &= (-1)^m f^{**} \left(\frac{(-1)^m \prod_{j=1}^{m} y_j^2}{e(\widetilde{V})} \right) [M]. \end{aligned}$$

[3]In the notation of Proposition 3.2 the x_j correspond to the ω_j and the y_j to the w_j.

The numerator of the fraction is precisely the Chern class $c_{2m}(\widetilde{V}\otimes\mathbb{C})$. This by definition equals the Pontryagin class $(-1)^m p_m(\widetilde{V})$ and one computes that $p_m(\widetilde{V}) = e(\widetilde{V})^2$, see [14]. Cancelling and applying the pullback we get a simple formula for the topological index of the de Rham complex

$$\mathrm{ind} = e(TM)[M].$$

This must equal the analytical index which we have computed to be the Euler characteristic of M. Hence we have obtained the well-known (see [17]) relation

$$\chi(M) = e(TM)[M]$$

between the Euler class and the Euler characteristic.

As a second example we will compute the index of the Salamon's complex (2.3) in the special case of manifolds admitting a $\mathrm{GL}(n,\mathbb{H})$-structure with a torsion-free connection. This applies, for example, to hyperkähler manifolds.

Example 5.6. Let M be a compact $4m$-dimensional manifold with a $\mathrm{GL}(m,\mathbb{H})$-structure admitting a torsion-free connection. Because $\mathrm{GL}(m,\mathbb{H})$ is a subgroup of $\mathrm{Sp}(1)\mathrm{GL}(m,\mathbb{H})$, the manifold M is quaternionic and so the Salamon's complex has sense. However, in this case both the bundles E and F exist globally and E is trivial. Moreover, the cotangent bundle T^*M is isomorphic to the complex vector bundle F up to the orientation – for m even the orientations coincide and for m odd they are opposite. Indeed, T^*M is oriented as a quaternionic vector bundle while F as a complex vector bundle. The representations $\mathbb{A}^j$ look like $\mathbb{C}^{j+1}\otimes\Lambda^j\mathbb{F}$.

By introducing a Riemannian metric on M we may reduce the $\mathrm{GL}(m,\mathbb{H})$-structure to a $\mathrm{Sp}(m)$-structure $\mathcal{P}$ (not necessarily admitting a torsion-free connection) and thus apply Proposition 5.4 with $\rho\colon\ \mathrm{Sp}(m)\hookrightarrow\mathrm{SO}(4m)$ being the standard inclusion from Section 2 and $\mathbb{E}^j=\mathbb{A}^j$. The Euler class of the universal bundle $\widetilde{V} = E\mathrm{Sp}(m)\times_\rho\mathbb{R}^{4m}$ equals the top-dimensional symplectic Pontryagin class (see [18]), which is nonzero, hence the condition is satisfied. Consider furthermore the universal vector bundles $\widetilde{F} = E\mathrm{Sp}(m)\times_{\mathrm{Sp}(m)}\mathbb{F}$ and $\widetilde{A^j} = E\mathrm{Sp}(m)\times_{\mathrm{Sp}(m)}\mathbb{A}^j$. Then $\widetilde{V}\cong\widetilde{F}$ up to the orientation and $\widetilde{A^j}\cong\mathbb{C}^{j+1}\otimes\Lambda^j\widetilde{F}$. By noting $\overline{\widetilde{F}}\cong\widetilde{F}$ this implies that we have

$$e(\widetilde{V}) = (-1)^m c_{2m}(\widetilde{F}),\quad \mathrm{td}(\widetilde{V}\otimes\mathbb{C}) = \mathrm{td}(\widetilde{F}\oplus\overline{\widetilde{F}}) = \mathrm{td}(\widetilde{F})^2.$$

Therefore, if $f\colon M\to B\mathrm{Sp}(m)$ is the classifying map for $\mathcal{P}$, then the index of the Salamon's complex is given by

$$\mathrm{ind} = f^{**}\left(\frac{\sum_{j=0}^{2m}(-1)^j(j+1)\mathrm{ch}\,\Lambda^j\widetilde{F}}{(-1)^m c_{2m}(\widetilde{F})}\cdot\mathrm{td}(\widetilde{F})^2\right)[M]. \tag{5.7}$$

To simplify this formula, we will first have to compute the Chern classes of the complex vector bundle $\widetilde{F}$ using Proposition 3.2. As a maximal torus S of the group $\mathrm{Sp}(m)$ we take the subgroup of diagonal matrices with entries $\exp(2\pi\mathrm{i}x_j)$, where $x_j\in\mathbb{R}$. Then $x_1, x_2, \dots, x_m$ viewed as linear forms on the Lie algebra $\mathfrak{s}$ constitute a basis of integral weights. The representation $\lambda\colon\ \mathrm{Sp}(m)\to\mathrm{U}(2m)$ will be the standard complex representation $\mathbb{F}$. The maximal torus S is mapped onto diagonal matrices with entries $\exp(\pm2\pi\mathrm{i}x_j)$, hence the weights of λ are the forms $\pm x_j, 1 \le j \le m$. Clearly, the λ-extension $E\mathrm{Sp}(m)_\lambda$ is precisely the principal frame bundle of the complex vector bundle $\widetilde{F}$. It follows that the total Chern class and the Todd class of $\widetilde{F}$ are

$$c(\widetilde{F}) = \prod_{j=1}^{m}(1+y_j)(1-y_j), \quad \mathrm{td}(\widetilde{F}) = \prod_{j=1}^{m} \frac{y_j(-y_j)}{(1-e^{-y_j})(1-e^{y_j})}.$$

In particular, the last Chern class is $c_{2m}(\widetilde{F}) = \prod_{j=1}^{m} y_j(-y_j)$.

It remains only to compute the numerator of the fraction in (5.7). Instead of the coefficient $(-1)^j$ write t^j and recall the formula (3.4). Then proceed as follows

$$\sum_{j=0}^{2m}(j+1)t^j \,\mathrm{ch}(\Lambda^j\widetilde{F}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=0}^{2m} t^{j+1}\,\mathrm{ch}(\Lambda^j\widetilde{F})\right) =$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\Big(t\cdot \mathrm{ch}(\Lambda_t(\widetilde{F})\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\left(t\prod_{j=1}^{m}(1+te^{y_j})(1+te^{-y_j})\right) =$$
$$= \prod_{j=1}^{m}(1+te^{y_j})(1+te^{-y_j}) + t\sum_{j=1}^{m}(e^{y_j}(1+te^{-y_j}) + e^{-y_j}(1+te^{y_j}))\prod_{\substack{k=1\\k\neq j}}^{m}(1+te^{y_k})(1+te^{-y_k}).$$

Substitute $t=-1$ and consider the first factor in the sum on the right

$$e^{y_j}(1-e^{-y_j}) + e^{-y_j}(1-e^{y_j}) = e^{y_j}(1-e^{-y_j}) - (1-e^{-y_j}) = -(1-e^{-y_j})(1-e^{y_j}).$$

Inserting back into the formula above and looking at the summands we end up with

$$\sum_{j=0}^{2m}(-1)^j(j+1)\mathrm{ch}\,\Lambda^j\widetilde{F} = (m+1)\prod_{j=1}^{m}(1-e^{y_j})(1-e^{-y_j}).$$

The interior of the bracket in (5.7) now can be written as

$$\frac{(m+1)\prod_{j=1}^{m}(1-e^{y_j})(1-e^{-y_j})}{(-1)^m\prod_{j=1}^{m}y_j(-y_j)}\cdot\left(\prod_{j=1}^{m}\frac{y_j(-y_j)}{(1-e^{-y_j})(1-e^{y_j})}\right)^2 = (-1)^m(m+1)\,\mathrm{td}(\widetilde{F}).$$

Applying the map f^{**} and evaluating on the fundamental class of M we obtain the desired index. The complex tangent bundle T^cM of M is isomorphic to the complex vector bundle $F^* \cong F$. Therefore, we have proved the following proposition, which is our first partial result on indices of quaternionic complexes.

Proposition 5.8. *Let M be a compact manifold with a* $\mathrm{GL}(m,\mathbb{H})$*-structure admitting a torsion-free connection. Then the index of the Salamon's complex is given by*

$$(-1)^m(m+1)\mathrm{td}(T^cM)[M].$$

Note that such a manifold is a complex manifold and the number $\mathrm{td}(T^cM)[M]$ is the index of the Dolbeault complex associated to the complex tangent bundle T^cM of M. In particular, it is an integer and thus the above index is an integer divisible by $m+1$.

We should remark here that by a different method very similar results were obtained in [6] for a certain class of quaternionic complexes. However, it is not clear to us whether the Salamon's complex is included in that class.

6. Quaternionic structures

In this section we will look more closely at some topological properties behind quaternionic structures. In particular, there are naturally defined characteristic classes for these structures, which will enable us to describe the rational cohomology ring of the classifying space $B\mathrm{Sp}(1)\mathrm{Sp}(m)$. Once we have done this, we will be able to compute the indices of quaternionc complexes we are interested in. The basic source for the notions and constructions presented here is [9].

Throughout the section, X will denote a compact Hausdorff topological space.

Definition 6.1. Let β be an oriented real 3-dimensional vector bundle over X with a positive-definite inner product $\langle -,-\rangle$. Then we define a *bundle of quaternion algebras* $\mathbb{H}_\beta = \mathbb{R}\oplus\beta$ with the multiplication given by

$$(s,u)\cdot(t,v) = (st - \langle u,v\rangle, sv + tu + u\times v).$$

Equivalently, if $\mathcal{P}\to X$ is the principal $\mathrm{SO}(3) = \mathrm{Aut}(\mathbb{H})$-bundle corresponding to β, then

$$\mathbb{H}_\beta = \mathcal{P}\times_{\mathrm{Aut}(\mathbb{H})}\mathbb{H} \quad \text{and} \quad \beta = \mathcal{P}\times_{\mathrm{SO}(3)}\mathbb{R}^3$$

The definition says that fibrewise the bundle $\mathbb{H}_\beta$ carries a structure of the algebra of quaternions, but globally it may not be the product bundle $X\times\mathbb{H}$.

Definition 6.2. Let $V\to X$ be a real vector bundle. We say that V is a *right* $\mathbb{H}_\beta$*-bundle* if it admits a right $\mathbb{H}_\beta$-module structure, i.e. there is a bundle map $V\otimes_{\mathbb{R}}\mathbb{H}_\beta\to V$ that restricts to an $\mathbb{H}$-module structure in each fibre.

It follows from the definition that the dimension of an $\mathbb{H}_\beta$-bundle must be divisible by four. Moreover, such a bundle can be canonically oriented. Indeed, to orient the fibre V_x, choose a basis $e_1, e_2, \ldots, e_m$ of V_x as an $(\mathbb{H}_\beta)_x$-module and an oriented orthonormal basis $\mathrm{i},\mathrm{j},\mathrm{k}$ of β_x. Then $e_1, e_1\mathrm{i}, e_1\mathrm{j}, e_1\mathrm{k}, \ldots, e_m, e_m\mathrm{i}, e_m\mathrm{j}, e_m\mathrm{k}$ is the oriented basis of V_x.

These notions are related to the theory of quaternionic manifolds and to our problem of computing indices of quaternionic complexes by the following statement.

Proposition 6.3 ([9])**.** *A $4m$-dimensional real vector bundle V is a right $\mathbb{H}_\beta$-bundle for some oriented 3-dimensional vector bundle β if and only if the structure group of the frame bundle of V may be reduced to the subgroup* $\mathrm{Sp}(1)\mathrm{Sp}(m)\subset\mathrm{GL}(4m,\mathbb{R})$.

Proof. If the structure group of the principal frame bundle $\mathrm{Fr}(V)$ reduces to the subgroup $G_0 = \mathrm{Sp}(1)\,\mathrm{Sp}(m)$, then $V = \mathrm{Fr}(V)\times_{G_0}\mathbb{H}^m$, here we view $\mathbb{H}^m$ as a real vector space. Put $\beta = \mathrm{Fr}(V)\times_{G_0}\mathrm{im}\,\mathbb{H}$ with the action of G_0 on $\mathrm{im}\,\mathbb{H}$ being defined as follows: if $(a,A)\in\mathrm{Sp}(1)\times\mathrm{Sp}(m)$ represents an element of G_0, then $(a,A)\cdot q = aq\bar{a}$. Clearly, β is an oriented 3-dimensional real vector bundle and the associated quaternion algebra is $\mathbb{H}_\beta = \mathrm{Fr}(V)\times_{G_0}\mathbb{H}$ with the same action of G_0 as above. But then right multiplication by quaternions is a G_0-map and so it induces a right $\mathbb{H}_\beta$-module structure on V.

For the other direction see [9]. ☐

The proposition applies, in particular, to the tangent and cotangent bundle of a quaternionic manifold M (after introducing a Riemannian metric). We can actually describe the vector bundle β as follows. Let $\mathbb{E}$ be the standard complex Sp(1)-module as in Section 2. By letting Sp(m) act on $\mathbb{E}$ trivially, we may view $\mathbb{E}$ as a representation of the group $G_0 = \mathrm{Sp}(1)\mathrm{Sp}(m)$. Now consider the symmetric power $S^2\mathbb{E}$ and a mapping $\varphi\colon \operatorname{im}\mathbb{H} \to S^2\mathbb{E}$ defined by $\varphi(u) = \mathrm{j}\otimes u - 1\otimes u\mathrm{j}$. Then φ is a real linear G_0-map. Moreover, the real basis $\mathrm{i},\mathrm{j},\mathrm{k}$ of $\operatorname{im}\mathbb{H}$ is mapped to a complex basis of $S^2\mathbb{E}$

$$\begin{aligned} \mathrm{i} &\mapsto (1\otimes \mathrm{j} + \mathrm{j}\otimes 1)\mathrm{i}, \\ \mathrm{j} &\mapsto 1\otimes 1 + \mathrm{j}\otimes \mathrm{j}, \\ \mathrm{k} &\mapsto (1\otimes 1 - \mathrm{j}\otimes \mathrm{j})\mathrm{i}. \end{aligned}$$

This implies that the complexification of $\operatorname{im}\mathbb{H}$ is isomorphic to $S^2\mathbb{E}$ and, on the vector bundle level, the complexification of $\beta = \mathcal{P}\times_{G_0}\operatorname{im}\mathbb{H}$ is isomorphic to S^2E, which is a globally defined vector bundle over M.[4]

We will proceed to define characteristic classes for $\mathbb{H}_\beta$-bundles. Let $V \to X$ be a right $\mathbb{H}_\beta$-bundle of quaternionic dimension m, i.e. real dimension $4m$. Then one can consider the associated projective bundle $\mathbb{H}_\beta P(V) \to X$ whose fibre over a point $x \in X$ is the space of all quaternionic lines in the fibre V_x in the sense of the $\mathbb{H}_\beta$-module structure. Futhermore, let $L = \{(\ell, v) \in \mathbb{H}_\beta P(V) \times V \mid v \in \ell\}$ be the canonical $\mathbb{H}_\beta$-line bundle over $\mathbb{H}_\beta P(V)$ oriented as a right $\mathbb{H}_\beta$-bundle. The following proposition defines characteristic classes $d_j^\beta(V)$ of the bundle V as coefficients of a certain polynomial over the ring $H^*(X;\mathbb{Z})$.

Proposition 6.4 ([9])**.** *For each right $\mathbb{H}_\beta$-bundle $V \to X$ of quaternionic dimension m there are uniquely determined classes $d_j^\beta(V) \in H^{4j}(X;\mathbb{Z}), 1 \le j \le m$, such that we have*

$$H^*(\mathbb{H}_\beta P(V);\mathbb{Z}) = H^*(X;\mathbb{Z})[t]/(t^m - d_1^\beta(V)t^{m-1} + \ldots + (-1)^m d_m^\beta(V)),$$

where $t = e(L) \in H^4(\mathbb{H}_\beta P(V);\mathbb{Z})$ is the Euler class of the canonical bundle L.

Proof. The proof goes along as for the Chern classes of complex vector bundles by applying the Leray-Hirsch theorem, see for example [14]. ◻

One can show (see [9]) that the cohomology classes $d_j^\beta(V)$ have usual properties of characteristic classes like naturality or multiplicativity, i.e. $d^\beta(V_1 \oplus V_2) = d^\beta(V_1)d^\beta(V_2)$, where $d^\beta(V) = 1 + d_1^\beta(V) + d_2^\beta(V) + \ldots + d_m^\beta(V)$. In particular, there is a splitting principle for $\mathbb{H}_\beta$-bundles.

Proposition 6.5 ([9])**.** *For each $\mathbb{H}_\beta$-bundle $V \to X$ there is a fibre bundle $p\colon F(V) \to X$ such that the pullback p^*V splits into a direct sum of $\mathbb{H}_\beta$-line bundles and, moreover, the induced map $p^*\colon H^*(X;\mathbb{Z}) \to H^*(F(V);\mathbb{Z})$ is injective.*

Proof. The proof is again classical as for the Chern classes, see [14]. ◻

The splitting principle implies that in calculations with the classes $d_j^\beta(V)$ we may formally assume that there are cohomology classes $y_1, y_2, \ldots, y_m \in H^4(X;\mathbb{Z})$ such that $d_j^\beta(V)$ is the j-th elementary symmetric polynomial in the y_k's or, in short, $d^\beta(V) = \prod_{k=1}^m (1 + y_k)$.

[4]Compare with ([22], page 146).

The following proposition shows that our classes determine other characteristic classes of V as a real vector bundle. This will be technically very useful.

Proposition 6.6 ([9]). *Let V be a canonically oriented right $\mathbb{H}_\beta$-bundle of quaternionic dimension m, i.e. of real dimension $4m$.*

(a) *The Euler class $e(V)$ of V equals the top-dimensional class $d_m^\beta(V)$.*

(b) *The rational Pontryagin classes of V are given by*

$$1 + p_1(V) + p_2(V) + \ldots + p_{2m}(V) = \prod_{j=1}^{m}((1 + y_j)^2 + p_1(\beta)),$$

where $d^\beta(V) = \prod_{j=1}^{m}(1 + y_j)$.

Proof. See [9], but note that we deal with right $\mathbb{H}_\beta$-bundles rather then left ones. □

The final point of this section is the rational cohomology ring of the classifying space BG_0 of the group $G_0 = \mathrm{Sp}(1)\mathrm{Sp}(m)$. Let EG_0 be the universal principal G_0-bundle and put $\beta = EG_0 \times_{G_0} \operatorname{im}\mathbb{H}$ and $\widetilde{V} = EG_0 \times_{G_0} \mathbb{R}^{4m}$. Furthermore, write q_1 for the first Pontryagin class $p_1(\beta)$ of β and d_j for the characteristic classes $d_j^\beta(\widetilde{V})$.

Proposition 6.7 ([9]). *The rational cohomology ring of $B\mathrm{Sp}(1)\mathrm{Sp}(m)$ is given by*

$$H^*(B\mathrm{Sp}(1)\mathrm{Sp}(m); \mathbb{Q}) \cong \mathbb{Q}[q_1, d_1, d_2, \ldots, d_m].$$

Proof. One can obtain this from the description of the integral cohomology ring of the classifying space $B\mathrm{Sp}(1)\mathrm{Sp}(m)$, which was done in [9]. □

According to Proposition 6.6 the Euler class $e(\widetilde{V})$ equals the class d_m, which is a generator of the cohomology ring and so it is nonzero. Hence we have verified the condition of Proposition 5.4 and may now proceed to calculations.

7. The computations

In this last section we will describe a procedure how to compute the indices of the quaternionic complexes from Theorem 2.11 and illustrate it on some examples.

Let M be a compact $4m$-dimensional quaternionic manifold. By introducing a Riemannian metric on M we may reduce the structure group of the principal frame bundle of M to the subgroup $G_0 = \mathrm{Sp}(1)\mathrm{Sp}(m)$. Let $\mathcal{P}$ be the corresponding principal G_0-bundle and $f \colon M \to BG_0$ the classifying map of $\mathcal{P}$. Put $\widetilde{W}_k^j = EG_0 \times_{G_0} \mathbb{W}_k^j$ and $\widetilde{V} = EG_0 \times_{G_0} \mathbb{R}^{4m}$. Then by Proposition 5.4 the index of the $D^{0,1}$-complex D_k associated to the representation $\mathbb{W}_k$ from Theorem 2.11 is given by

$$\operatorname{ind} D_k = \left\{ f^{**} \left(\frac{\sum_{j=0}^{2m}(-1)^j \operatorname{ch} \widetilde{W}_k^j}{e(\widetilde{V})} \right) \cdot \operatorname{td}(TM \otimes \mathbb{C}) \right\} [M]. \tag{7.1}$$

To evaluate this formula we will have to solve the equation

$$x \cup e(\widetilde{V}) = \sum_{j=0}^{2m} (-1)^j \operatorname{ch} \widetilde{W}_k^j \tag{7.2}$$

in the ring $H^{**}(BG_0;\mathbb{Q})$ to determine the fraction above. This problem can be simplified it two ways. First, because the cohomology groups of the compact manifold M are zero from dimension $4m+1$, it suffices to compute x up to dimension $4m$. Secondly, if BG_1 is the classifying space of the group $G_1 = \mathrm{Sp}(1)\times\mathrm{Sp}(m)$ and $\pi\colon G_1\to G_0$ the projection, then the induced map $(B\pi)^*\colon H^*(BG_0;\mathbb{Q})\to H^*(BG_1;\mathbb{Q})$ is an isomorphism. Therefore, we may pull back the equation (7.2) to BG_1 and solve it in $H^*(BG_1;\mathbb{Q})$.

Let again $\mathbb{E}$ and $\mathbb{F}$ denote the standard complex $\mathrm{Sp}(1)$-module and $\mathrm{Sp}(m)$-module, respectively. By letting the other factor of $G_1 = \mathrm{Sp}(1)\times\mathrm{Sp}(m)$ act trivially, we may view these modules as representations of G_1. Put $\widetilde{E} = EG_1\times_{G_1}\mathbb{E}$ and $\widetilde{F} = EG_1\times_{G_1}\mathbb{F}$. Then these are globally defined vector bundles over BG_1 and we have

$$(B\pi)^*(\widetilde{V}\otimes_{\mathbb{R}}\mathbb{C})\cong\widetilde{E}\otimes_{\mathbb{C}}\widetilde{F} \tag{7.3}$$

as in Section 2. Moreover, if $\beta = EG_0\times_{G_0}\operatorname{im}\mathbb{H}$, then the real vector bundle $\widetilde{V}_1 = (B\pi)^*(\widetilde{V})$ is a right $\mathbb{H}_{\beta_1}$-bundle for $\beta_1 = (B\pi)^*(\beta)$. Similarly as in the preceding section we get $\beta_1\otimes\mathbb{C}\cong S^2\widetilde{E}$ and so for the first Pontryagin class of β_1 we have

$$p_1(\beta_1) = -c_2(S^2\widetilde{E}) = -4c_2(\widetilde{E}). \tag{7.4}$$

The second equality will be proved later.

From the isomorphism (7.3) we obtain

$$\begin{aligned}4m + p_1(\widetilde{V}_1) + \frac{1}{12}p_1(\widetilde{V}_1)^2 - \frac{1}{6}p_2(\widetilde{V}_1) + \ldots &= \operatorname{ch}(\widetilde{V}_1\otimes\mathbb{C}) = \operatorname{ch}(\widetilde{E})\operatorname{ch}(\widetilde{F}) = \\ &= \left(2 + c_1(\widetilde{E}) + \frac{1}{2}c_1(\widetilde{E})^2 - c_2(\widetilde{E}) + \ldots\right)\left(2m + c_1(\widetilde{F}) + \frac{1}{2}c_1(\widetilde{F})^2 - c_2(\widetilde{F})\ldots\right).\end{aligned}$$

By comparing inductively the two sides of this equality we may write the Pontryagin classes of $\widetilde{V}_1$ as polynomials in the Chern classes of $\widetilde{E}$ and $\widetilde{F}$. This implies together with (7.4) and Proposition 6.6 that we are able to translate between three sets of characteristic classes – the Chern classes of $\widetilde{E}$ and $\widetilde{F}$, the Pontryagin classes of $\widetilde{V}_1$ and β_1 and the quaternionic classes $d_1^{\beta_1}(\widetilde{V}_1), d_2^{\beta_1}(\widetilde{V}_1),\ldots,d_m^{\beta_1}(\widetilde{V}_1)$ and $p_1(\beta_1)$.

Now return to the equation (7.2) in the pulled back version, i.e.

$$(B\pi)^*(x)\cup e(\widetilde{V}_1) = \sum_{j=0}^{2m}(-1)^j\operatorname{ch}(B\pi)^*(\widetilde{W_k^j}). \tag{7.5}$$

To solve this equation we have to compute first the Chern characters in terms of the Chern classes of $\widetilde{E}$ and $\widetilde{F}$ and then express these in the $d_l^{\beta_1}$-classes. Afterwards, we divide by the Euler class $e(\widetilde{V}_1) = d_m^{\beta_1}(\widetilde{V}_1)$ to obtain the solution $(B\pi)^*(x)\in H^*(BG_1;\mathbb{Q})$, which can be sent back to $H^*(BG_0;\mathbb{Q})$.

Let us focus on the bundles $(B\pi)^*(\widetilde{W_k^j}) = EG_1\times_{G_1}\mathbb{W}_k^j$. We know from (2.10) that

$$(B\pi)^*(\widetilde{W_k^j})\cong S^{j+k}\widetilde{E}\otimes(\Lambda^j\widetilde{F}\otimes S^k\widetilde{F}^*)_0\quad\text{for } j<2m,\qquad (B\pi)^*(\widetilde{W_k^{2m}}) = S^{2(m+k)}\widetilde{E}\otimes\Lambda^{2m}\widetilde{F}.$$

The factors in the tensor products are globally defined vector bundles and so to compute the Chern character of $(B\pi)^*(\widetilde{W_k^j})$ it suffices to compute the Chern characters of the factors.

We will apply Proposition 3.2. As a maximal torus S of $G_1 = \mathrm{Sp}(1)\times\mathrm{Sp}(m)$ take the product of the standard maximal tori of $\mathrm{Sp}(1)$ and $\mathrm{Sp}(m)$ – the standard maximal torus of Sp(1) is the set of complex units $\exp(2\pi\mathrm{i}x)$ and the standard maximal torus of $\mathrm{Sp}(m)$ is the set of diagonal matrices with entries $\exp(2\pi\mathrm{i}x_l)$, where $x_l \in \mathbb{R}$. Then $x, x_1, x_2, \ldots, x_m$ viewed as linear forms on the Lie algebra $\mathfrak{s}$ constitute a basis of integral weights.

Consider first the complex vector bundle $\widetilde{E} = EG_1 \times_{G_1} \mathbb{E}$. The inducing representation of this bundle is $\lambda\colon G_1 = \mathrm{Sp}(1) \times \mathrm{Sp}(m) \to \mathrm{U}(\mathbb{E}) \cong \mathrm{U}(2)$, the projection onto the first factor composed with the standard representation of Sp(1) on $\mathbb{E}$. An element of the maximal torus S is mapped to a diagonal matrix of the form

$$\begin{pmatrix} \exp(2\pi\mathrm{i}x) & 0 \\ 0 & \exp(-2\pi\mathrm{i}x) \end{pmatrix}$$

and so the weights of λ are the forms $\pm x$. Then the Chern classes of $\widetilde{E}$ are given by[5]

$$c(\widetilde{E}) = (1+y)(1-y) \Rightarrow c_1(\widetilde{E}) = 0,\ c_2(\widetilde{E}) = -y^2.$$

The Chern classes of the symmetric powers $S^j\widetilde{E}$ are now easy to compute. The map λ is again the projection onto the first factor of G_1 composed with the standard representation of Sp(1) on $S^j\mathbb{E}$. If e_1, e_2 is the canonical basis of $\mathbb{E}$, then the symmetric products $e_1^{k_1} e_2^{k_2}$, $k_1 + k_2 = j$ form a basis of $S^j\mathbb{E}$ and the action of an element $s = \exp(2\pi\mathrm{i}x) \in S$ of the maximal torus of G_1 is given by

$$\lambda(s)(e_1^{k_1} e_2^{k_2}) = \exp[(2\pi\mathrm{i})(k_1 - k_2)x] \cdot e_1^{k_1} e_2^{k_2}.$$

Therefore, the weights of λ are the forms $(k_1 - k_2)x$, $k_1 + k_2 = j$, and we have

$$c(S^j\widetilde{E}) = \prod_{k_1+k_2=j} (1 + (k_1 - k_2)y).$$

We see from this that the Chern classes of $S^j\widetilde{E}$ can be written as polynomials in $c_2(\widetilde{E})$. In particular, setting $j = 2$ we obtain

$$c(S^2\widetilde{E}) = (1+2y)(1-2y) = 1 - 4y^2 = 1 + 4c_2(\widetilde{E}),$$

which proves the second equality in (7.4).

Now turn to the vector bundle $\widetilde{F}$. The inducing representation $\lambda\colon G_1 \to \mathrm{U}(\mathbb{F}) \cong \mathrm{U}(2m)$ is the projection onto the second factor composed with the standard representation of $\mathrm{Sp}(m)$ on $\mathbb{F}$. If $e_1, e_2, \ldots, e_{2m-1}, e_{2m}$ is the canonical basis of $\mathbb{F}$, then the action of an element $s \in S$ of the maximal torus of G_1 is given by

$$\lambda(s)(e_{2l}) = \exp(2\pi\mathrm{i}x_l) \cdot e_{2l}, \quad \lambda(s)(e_{2l+1}) = \exp(-2\pi\mathrm{i}x_l) \cdot e_{2l+1}.$$

It follows that the weights of λ are $\pm x_l$, $1 \le l \le m$, and then the total Chern class of $\widetilde{F}$ is

$$c(\widetilde{F}) = \prod_{l=1}^{m} (1 + y_l)(1 - y_l) = \prod_{l=1}^{m} (1 - y_l^2).$$

The Chern classes of the exterior powers $\Lambda^j\widetilde{F}$ may be obtained as follows. In this case the representation λ is the projection onto the second factor of G_1 composed with the standard representation of $\mathrm{Sp}(m)$ on $\Lambda^j\mathbb{F}$. If we put $x'_{2l} = x_l$ and $x'_{2l+1} = -x_l$, $0 \le l \le m$,

[5]As before, y stands for the transgression of x and y_l will stand for the transgressions of x_l. Moreover, we omit the map η^* according to the paragraph following Proposition 3.2.

then the weights of λ are the sums $x'_{l_1} + x'_{l_2} + \ldots + x'_{l_j}$, where $1 \le l_1 < l_2 < \ldots < l_j \le 2m$ (see Example 3.3). The total Chern class of $\Lambda^j \widetilde{F}$ is then given by

$$c(\Lambda^j \widetilde{F}) = \prod_{1 \le l_1 < \ldots < l_j \le 2m} (1 + y'_{l_1} + \ldots + y'_{l_j}).$$

Substituting back $y'_{2l} = y_l$ and $y'_{2l+1} = -y_l$ we obtain an expression symmetric in $-y_l^2$ and so it can be written as a polynomial in the Chern classes of $\widetilde{F}$.

The computation of the Chern classes of $(\Lambda^j \widetilde{F} \otimes S^k \widetilde{F}^*)_0$ for $k > 0$ is much more complicated. The representation λ is given by the projection onto the first factor followed by the action of $\mathrm{Sp}(m)$ on $(\Lambda^j \mathbb{F} \otimes S^k \mathbb{F}^*)_0$. However, $\mathbb{V}^j_k = (\Lambda^j \mathbb{F} \otimes S^k \mathbb{F}^*)_0$ is defined as a representation of the group $\mathrm{U}(2m)$ corresponding to a maximal weight and we have to find its weights with respect to the subgroup $\mathrm{Sp}(m) \subset \mathrm{U}(2m)$. This can be done as follows. First, the character ring of complex representations of the group $\mathrm{SU}(2m)$ differs from that of the group $\mathrm{U}(2m)$ only by a one-dimensional determinantal representation on which $\mathrm{Sp}(m)$ acts trivially. Therefore, there is nothing lost in assuming that $\mathbb{V}^j_k$ is a representation of $\mathrm{SU}(2m)$. But $\mathrm{SU}(2m)$ is compact and simply connected and so its representation theory is equivalent to that of the complex Lie algebra $\mathfrak{sl}(2m, \mathbb{C})$. In particular, there are algorithms for computing all the weights of $\mathbb{V}^j_k$ if we know its maximal weight.[6] This maximal weight is by definition the sum of the maximal weight of $\Lambda^j \mathbb{F}$ and the maximal weight of $S^k \mathbb{F}^*$ and these are easy to find – if $z_1, z_2, \ldots, z_{2m}$ are the standard integral weights for $\mathrm{SU}(2m)$, then the maximal weight of $\Lambda^j \mathbb{F}$ is $z_1 + z_2 + \ldots + z_j$ while the maximal weight of $S^k \mathbb{F}^*$ is $-k \cdot z_{2m}$. The weights of $\mathbb{V}^j_k$ then will be integral linear combinations of the z_l's. To obtain the weights of $\mathbb{V}^j_k$ as a representation of $\mathrm{Sp}(m)$ we only have to substitute $z_{2l} = x_l$ and $z_{2l+1} = -x_l$ for $1 \le l \le m$ – this can be seen from the definition of the standard inclusion $\mathrm{Sp}(m) \subset \mathrm{SU}(2m)$. Finally, once we know the weights, we may compute the total Chern class $c(\mathbb{V}^j_k)$, which will be an expression symmetric in $-y_l^2$. Indeed, the set of weights of a representation is invariant under the action of the Weyl group and the Weyl group of $\mathrm{SU}(2m)$ is the symmetry group on the set $\{z_1, y_2, \ldots, z_{2m}\}$. Therefore, $c(\mathbb{V}^j_k)$ can be again expressed as a polynomial in the Chern classes of $\widetilde{F}$.

To conclude, we have seen that the Chern classes of $S^{j+k} \widetilde{E}$ and $(\Lambda^j \widetilde{F} \otimes S^k \widetilde{F}^*)_0$ can be written as polynomials in the Chern classes of $\widetilde{E}$ and $\widetilde{F}$, respectively. Then this is also true for the Chern characters of these bundles and the bundle $(B\pi)^*(\widetilde{W}^j_k)$. The right-hand side of (7.5) is thus a polynomial in the Chern classes of $\widetilde{E}$ and $\widetilde{F}$ and so can be expressed in terms of the quaternionic classes $d^{\beta_1}_l(\widetilde{V}_1)$. The result will be a multiple of the Euler class $e(\widetilde{V}_1) = d^{\beta_1}_m(\widetilde{V}_1)$. Dividing by $e(\widetilde{V}_1)$, we obtain the solution $(B\pi)^*(x) \in H^*(BG_1; \mathbb{Q})$. To get $x \in H^*(BG_0; \mathbb{Q})$, it suffices to write $d^{\beta}_l(\widetilde{V})$ and $p_1(\beta)$ instead of $d^{\beta_1}_l(\widetilde{V}_1)$ and $p_1(\beta_1)$.

The computation is almost finished. Recall that x was the fraction in (7.1). If we express x in terms of the Pontryagin classes of $\widetilde{V}$ and $p_1(\beta)$, then $f^{**}(x)$ will be a polynomial in the Pontryagin classes of TM and the class $p_1(f^*\beta)$. Note that $\mathbb{H}_{f^*\beta}$ is the bundle of quaternion algebras corresponding to the G_0-structure $\mathcal{P}$ on M. The final step is to multiply with

[6] We have used the computer algebra system LiE [24].

the Todd class $\mathrm{td}(TM \otimes \mathbb{C})$ and evaluate the top-dimensional part of the product on the fundamental class $[M]$ of M.

We will go through the computation in the easiest case of the Salamon's complex on an 8-dimensional quaternionic manifold, i.e. $m = 2$ and $k = 0$.

Example 7.6. Let M be a compact quaternionic manifold M of dimension 8 and consider the Salamon's complex (2.3). We have to deal with the vector bundles

$$(B\pi)^*(\widetilde{W}_0^j) = S^j\widetilde{E} \otimes \Lambda^j\widetilde{F}, \quad 0 \le j \le 4.$$

For simplicity we make the following notation

$$p_j(\widetilde{V}_1) = p_j \text{ for } 1 \le j \le 4, \quad p_1(\beta_1) = q_1, \quad d_1^{\beta_1}(\widetilde{V}_1) = d_1, \quad d_2^{\beta_1}(\widetilde{V}_1) = d_2,$$
$$c_2(\widetilde{E}) = a_2, \quad c_2(\widetilde{F}) = b_2, \quad c_4(\widetilde{F}) = b_4.$$

Note that we have seen that the odd-dimensional Chern classes of $\widetilde{E}$ and $\widetilde{F}$ are zero.

1. The first preliminary step of the computation is to write out the relations between the above characteristic classes. By Proposition 6.6 we have

$$1 + p_1 + p_2 + p_3 + p_4 = [(1 + y_1)^2 + q_1][(1 + y_2)^2 + q_1],$$

where $d_1 = y_1 + y_2$ and $d_2 = y_1y_2$. Expanding the right-hand side we obtain

$$p_1 = 2d_1 + 2q_1, \quad p_2 = d_1^2 + 2d_1q_1 + 2d_2 + q_1^2, \quad p_3 = d_1^2q_1 + 2d_1d_2 - 2d_2q_1, \quad p_4 = d_2^2,$$
$$d_1 = \frac{1}{2}p_1 - q_1, \quad d_2 = -\frac{1}{8}p_1^2 + \frac{1}{2}p_2.$$

Furthermore, the equality $\mathrm{ch}(\widetilde{V}_1 \otimes \mathbb{C}) = \mathrm{ch}(\widetilde{E})\,\mathrm{ch}(\widetilde{F})$ reads as

$$8 + p_1 + \frac{1}{12}p_1^2 - \frac{1}{6}p_2 + \ldots = \left(2 - a_2 + \frac{1}{12}a_2^2 + \ldots\right)\left(4 - b_2 + \frac{1}{12}b_2^2 - \frac{1}{6}b_4 + \ldots\right)$$

and so for the Pontryagin classes we have

$$p_1 = -4a_2 - 2b_2, \quad p_2 = 6a_2^2 + 2a_2b_2 + b_2^2 + 2b_4.$$

Finally, comparing this with the equalities above and from (7.4), we obtain

$$q_1 = -4a_2, \quad d_1 = 2a_2 - b_2, \quad d_2 = a_2^2 - a_2b_2 + b_4,$$
$$a_2 = -\frac{1}{4}q_1, \quad b_2 = -d_1 - \frac{1}{2}q_1, \quad b_4 = \frac{1}{4}d_1q_1 + d_2 + \frac{1}{16}q_1^2.$$

2. Now we will compute the Chern classes of the symmetric powers $S^j\widetilde{E}$ and the exterior powers $\Lambda^j\widetilde{F}$ in terms of the Chern classes of $\widetilde{E}$ and $\widetilde{F}$. First, we know that

$$c(\widetilde{E}) = (1 + y)(1 - y) = 1 - y^2 \Rightarrow a_2 = -y^2.$$

Then one easily obtains

$$c(S^2\widetilde{E}) = (1 + 2y)(1 - 2y) = 1 - 4y^2 = 1 + 4a_2,$$
$$c(S^3\widetilde{E}) = (1 + 3y)(1 + y)(1 - y)(1 - 3y) = 1 - 10y^2 + 9y^4 = 1 + 10a_2^2 + 9a_2^4,$$
$$c(S^4\widetilde{E}) = (1 + 4y)(1 + 2y)(1 - 2y)(1 - 4y) = 1 - 20y^2 + 64y^4 = 1 + 20a_2^2 + 64a_2^4.$$

Similarly, for the vector bundle $\widetilde{F}$ we have

$$c(\widetilde{F}) = (1+y_1)(1-y_1)(1+y_2)(1-y_2) = 1 - y_1^2 - y_2^2 + y_1^2y_2^2 \Rightarrow b_2 = -y_1^2 - y_2^2,\ b_4 = y_1^2y_2^2$$

and therefore

$$\begin{aligned} c(\Lambda^2\widetilde{F}) &= (1-y_1+y_2)(1-y_1-y_2)(1+y_1+y_2)(1+y_1-y_2) = \\ &= 1 - 2y_1^2 - 2y_2^2 + y_1^4 - 2y_1^2y_2^2 + y_2^4 = 1 + 2b_2 + b_2^4 - 4b_4, \\ c(\Lambda^3\widetilde{F}) &= (1+y_2)(1-y_2)(1+y_1)(1-y_1) = 1 + b_2 + b_4, \\ c(\Lambda^4\widetilde{F}) &= 1. \end{aligned}$$

3. Having computed the Chern classes of $S^j\widetilde{E}$ and $\Lambda^j\widetilde{F}$ we may evaluate the right-hand side of (7.5), i.e. the alternating sum $\sum_{j=0}^4 (-1)^j \operatorname{ch}(B\pi)^*(\widetilde{W_0^j}) = \sum_{j=0}^4 (-1)^j \operatorname{ch}(S^j\widetilde{E})\operatorname{ch}(\Lambda^j\widetilde{F})$. It suffices to compute the Chern characters up to dimension 16, because once we divide by the Euler class, the dimension decreases to 8, which is exactly the dimension of the manifold M. Unfortunately, these calculations are not so easy to handle and so we have used the computer algebra system Maple. The result is the following formula

$$\begin{aligned} \sum_{j=0}^4 (-1)^j \operatorname{ch}(B\pi)^*(\widetilde{W_0^j}) &= 3a_2^2 - 3a_2b_2 + 3b_4 - \frac{9}{2}a_2^3 + \frac{17}{4}a_2^2b_2 + \frac{1}{4}a_2b_2^2 - \frac{9}{2}a_2b_4 - \frac{1}{4}b_2b_4 + \\ &+ \frac{163}{80}a_2^4 - \frac{67}{40}a_2^3b_2 - \frac{17}{48}a_2^2b_2^2 + \frac{49}{24}a_2^2b_4 - \frac{1}{120}a_2b_2^3 + \frac{43}{120}a_2b_2b_4 + \frac{1}{120}b_2^2b_4 + \frac{1}{240}b_4^2. \end{aligned}$$

4. The next step is to express the formula in terms of the classes d_1, d_2 and q_1. As we know, we should get a multiple of the class d_2. Really, we have

$$\sum_{j=0}^4 (-1)^j \operatorname{ch}(B\pi)^*(\widetilde{W_0^j}) = 3d_2 + \frac{1}{4}d_1d_2 + \frac{5}{4}d_2q_1 + \frac{1}{120}d_1^2d_2 + \frac{1}{10}d_1d_2q_1 + \frac{1}{240}d_2^2 + \frac{7}{40}d_2q_1^2.$$

Now we divide by the Euler class $e(\widetilde{V_1}) = d_2$ to obtain the solution of the equation (7.5) and then express this in terms of the Pontryagin classes

$$\begin{aligned} (B\pi)^*(x) &= 3 + \frac{1}{4}d_1 + \frac{5}{4}q_1 + \frac{1}{120}d_1^2 + \frac{1}{10}d_1q_1 + \frac{1}{240}d_2 + \frac{7}{40}q_1^2 = \\ &= 3 + \frac{1}{8}p_1 + q_1 + \frac{1}{640}p_1^2 + \frac{1}{24}p_1q_1 + \frac{1}{480}p_2 + \frac{1}{12}q_1^2. \end{aligned}$$

5. Finally, we have to pull back the solution x to the manifold M and multiply by the Todd class $\operatorname{td}(TM \otimes \mathbb{C})$. The formulas for these two classes are

$$\begin{aligned} f^*(x) &= 3 + \frac{1}{8}p_1(TM) + p_1(f^*\beta) + \frac{1}{640}p_1(TM)^2 + \frac{1}{24}p_1(TM)p_1(f^*\beta) + \\ &\quad + \frac{1}{480}p_2(TM) + \frac{1}{12}p_1(f^*\beta)^2, \\ \operatorname{td}(TM \otimes \mathbb{C}) &= 1 - \frac{1}{12}p_1(TM) + \frac{1}{240}p_1(TM)^2 - \frac{1}{720}p_2(TM). \end{aligned}$$

The index of the Salamon's complex is then given by evaluating the top-dimensional part of the product $f^*(x) \cdot \mathrm{td}(TM \otimes \mathbb{C})$ on the fundamental class $[M]$. Hence,

$$\operatorname{ind} D_0 = \left(\frac{7}{1920} p_1(TM)^2 - \frac{1}{24} p_1(TM) p_1(f^*\beta) - \frac{1}{480} p_2(TM) + \frac{1}{12} p_1(f^*\beta)^2 \right) [M]$$

and this is the desired result of the computation.

Note that the formula depends on the G_0-structure on M via the characteristic class $p_1(f^*\beta)$. This class may be also expressed without any reference to the classifying map f. Indeed, we know that $\beta \otimes \mathbb{C} \cong S^2\widetilde{E}$, where $S^2\widetilde{E}$ is now viewed as a vector bundle over BG_0. But then $f^*(\beta \otimes \mathbb{C}) \cong S^2 E$ is a globally defined complex vector bundle over M and

$$p_1(f^*\beta) = -c_2(f^*(\beta \otimes \mathbb{C})) = -c_2(S^2 E).$$

In general, the basic computational problem is to find the weights of the representations $(\Lambda^j \mathbb{F} \otimes S^k \mathbb{F}^*)_0$ and then process these to obtain the Chern classes of the vector bundles $(\Lambda^j \widetilde{F} \otimes S^k \widetilde{F}^*)_0$. As was said before, one can make use of computer algebra systems such as LiE (see [24]) and Maple. We have carried out some calculations for 8 and 12-dimensional manifolds arriving at the following formulas.

Theorem 7.7. *Let M be an 8-dimensional compact quaternionic manifold. If we write $p_1 = p_1(TM)$, $p_2 = p_2(TM)$ and $q_1 = -c_2(S^2E)$, then we have*

$$\operatorname{ind} D_0 = \left(\frac{7}{1920} p_1^2 - \frac{1}{24} p_1 q_1 - \frac{1}{480} p_2 + \frac{1}{12} q_1^2 \right) [M],$$
$$\operatorname{ind} D_1 = \left(\frac{209}{1920} p_1^2 + \frac{11}{24} p_1 q_1 - \frac{167}{480} p_2 + \frac{25}{12} q_1^2 \right) [M].$$

Theorem 7.8. *Let M be a 12-dimensional compact quaternionic manifold. If we write $p_1 = p_1(TM)$, $p_2 = p_2(TM)$, $p_3 = p_3(TM)$ and $q_1 = -c_2(S^2E)$, then we have*

$$\operatorname{ind} D_0 = \Bigg(\frac{31}{241920} p_1^3 - \frac{7}{2304} p_1^2 q_1 - \frac{11}{60480} p_1 p_2 + \frac{41}{2304} p_1 q_1^2 + \\ + \frac{1}{576} p_2 q_1 + \frac{1}{15120} p_3 - \frac{73}{2304} q_1^3 \Bigg) [M],$$

$$\operatorname{ind} D_1 = \left(-\frac{1}{6720} p_1^3 - \frac{77}{576} p_1^2 q_1 + \frac{1}{280} p_1 p_2 - \frac{35}{576} p_1 q_1^2 + \frac{7}{18} p_2 q_1 - \frac{17}{840} p_3 - \frac{623}{576} q_1^3 \right) [M].$$

As an example we will evaluate the formulas for the quaternionic projective space and verify that the result is really an integer.

Example 7.9. Consider the quaternionic projective space $\mathbb{HP}^m$. In this case the bundles E and F from (2.2) exist globally and E is precisely the tautological line bundle. The cohomology ring $H^*(\mathbb{HP}^m; \mathbb{Z})$ is generated by the class $u = -c_2(E)$, which also satisfies $(u^m)[\mathbb{HP}^m] = 1$. Furthermore, one can show (see [8]) that the Pontryagin classes of $T\mathbb{HP}^m$ are given by

$$p(T\mathbb{HP}^m) = (1+u)^{2m+2}(1+4u)^{-1},$$

where $(1+4u)^{-1}$ is the inverse formal power series to $1+4u$. Finally, according to (7.4) we have $q_1 = -4c_2(E) = 4u$.

Let now $m = 2$. Then the Pontryagin classes are $p_1(T\mathbb{H}\mathrm{P}^2) = 2u$ and $p_2(T\mathbb{H}\mathrm{P}^2) = 7u^2$ and we may evaluate the two formulas from above

$$\operatorname{ind} D_0 = \left(\frac{7}{1920}4u^2 - \frac{1}{24}8u^2 - \frac{1}{480}7u^2 + \frac{1}{12}16u^2\right)[\mathbb{H}\mathrm{P}^2] = (u^2)\,[\mathbb{H}\mathrm{P}^2] = 1,$$
$$\operatorname{ind} D_1 = \left(\frac{209}{1920}4u^2 + \frac{11}{24}8u^2 - \frac{167}{480}7u^2 + \frac{25}{12}16u^2\right)[\mathbb{H}\mathrm{P}^2] = (35u^2)[\mathbb{H}\mathrm{P}^2] = 35.$$

Similarly, for $m = 3$ we have $p_1(T\mathbb{H}\mathrm{P}^3) = 4u$, $p_2(T\mathbb{H}\mathrm{P}^3) = 12u^2$, $p_3(T\mathbb{H}\mathrm{P}^3) = 8u^3$ and

$$\begin{aligned}\operatorname{ind} D_0 = \Big(&\frac{31}{241920}64u^3 - \frac{7}{2304}64u^3 - \frac{11}{60480}48u^3 + \frac{41}{2304}64u^3 + \\ &+ \frac{1}{576}48u^3 + \frac{1}{15120}8u^3 - \frac{73}{2304}64u^3\Big)[\mathbb{H}\mathrm{P}^3] = (-u^3)[\mathbb{H}\mathrm{P}^3] = -1,\\ \operatorname{ind} D_1 = \Big(&-\frac{1}{6720}64u^3 - \frac{77}{576}64u^3 + \frac{1}{280}48u^3 - \frac{35}{576}64u^3 + \\ &+ \frac{7}{18}48u^3 - \frac{17}{840}8u^3 - \frac{623}{576}64u^3\Big)[\mathbb{H}\mathrm{P}^3] = (-63u^3)[\mathbb{H}\mathrm{P}^3] = -63.\end{aligned}$$

The drawback of the index formulas is that they depend on the class q_1, which is not easy to compute. By taking integral linear combinations of the formulas we may try to eliminate the terms containing q_1 and so obtain some integrality conditions on the Pontryagin classes of the manifold. Consider for example the formulas from Theorem 7.7. We have

$$11\cdot\operatorname{ind} D_0 + \operatorname{ind} D_1 = \left(\frac{143}{960}p_1^2 - \frac{89}{240}p_2 + 3q_1^2\right)[M],$$
$$50\cdot\operatorname{ind} D_0 - 2\cdot\operatorname{ind} D_1 = \left(-\frac{17}{480}p_1^2 - 3p_1q_1 + \frac{71}{120}p_2\right)[M].$$

Recall that p_1 and q_1 are Pontryagin classes of some vector bundles and so they belong to the integral cohomology groups. In particular, evaluating p_1q_1 and q_1^2 on the fundamental class of M we get an integer. But then by evaluating the rest of the above formulas we must again obtain an integer (and not only a rational number).

Corollary 7.10. *Let M be an 8-dimensional compact quaternionic manifold. Then the following expressions are integers*

$$\left(\frac{143}{960}p_1^2 - \frac{89}{240}p_2\right)[M], \quad \left(-\frac{17}{480}p_1^2 + \frac{71}{120}p_2\right)[M].$$

Of course we can deal with many other integral linear combinations $a\cdot\operatorname{ind} D_0 + b\cdot\operatorname{ind} D_1$. The requirement is that the coefficients of the terms containing q_1 will be integers. This will hold true if and only if

$$-a + 11b \equiv 0 \mod 24, \quad a + 25b \equiv 0 \mod 12.$$

The second congruence is a consequence of the first and so for each solution

$$(a = 11b + 24k, b), \quad b, k \in \mathbb{Z}$$

of the first congruence we obtain another integrality conditions on the Pontryagin classes of the manifold.

Finally, we will look more closely on manifolds admitting a $\mathrm{GL}(m, \mathbb{H})$-structure with a torsion-free connection as in Example 5.6. In this case the formulas simplify considerably because the vector bundle E is trivial and so $q_1 = 0$. Furthermore, the complex tangent bundle T^cM of M is isomorphic to the vector bundle F up to the orientation.

Suppose that $m = 2$. Then the orientations coincide and as in Example 7.6 we may compute that $p_1(TM) = -2c_2(F)$ and $p_2(TM) = c_2(F)^2 + 2c_4(E)$. Now substitute to the first formula from Theorem 7.7

$$\operatorname{ind} D_0 = \left(\frac{7}{1920} 4c_2(F)^2 - \frac{1}{480}\left(c_2(F)^2 + 2c_4(F)\right)\right)[M] = \left(\frac{1}{80}c_2(F)^2 - \frac{1}{240}c_4(F)\right)[M].$$

The Todd class of F is given by

$$\mathrm{td}(F) = 1 + \frac{1}{12}c_2(F) + \frac{1}{240}c_2(F)^2 - \frac{1}{720}c_4(F).$$

In particular, three times the top dimensional part equals $\frac{1}{80}c_2(F)^2 - \frac{1}{240}c_4(F)$, which verifies that our general formula for the Salamon's complex from Theorem 7.7 coincides with the formula in Proposition 5.8.

Let us do the same for the second index formula from Theorem 7.7, i.e.

$$\operatorname{ind} D_1 = \left(\frac{209}{1920} 4c_2(F)^2 - \frac{167}{480}\left(c_2(F)^2 + 2c_4(F)\right)\right)[M] = \left(\frac{7}{80}c_2(F)^2 - \frac{167}{240}c_4(F)\right)[M] =$$

$$= 21\left(\frac{1}{240}c_2(F)^2 - \frac{1}{720}c_4\right)[M] - \frac{2}{3}c_4(F)[M] = 21\,\mathrm{td}(T^cM)[M] - \frac{2}{3}\chi(M).$$

The last equality follows from the facts that $\mathrm{td}(F) = \mathrm{td}(T^cM)$ and $c_4(F) = e(TM)$, the Euler class. Recall that the number $\mathrm{td}(T^cM)[M]$ must be an integer because it is the index of the Dolbeault complex on M, see [5]. Since the index $\operatorname{ind} D_1$ is also an integer, this implies that the Euler characteristic must be divisible by 3.

Corollary 7.11. *Let M be an 8-dimensional compact manifold with a $\mathrm{GL}(2, \mathbb{H})$-structure admitting a torsion-free connection. Then its Euler characteristic $\chi(M)$ is divisible by 3.*

This result is well-known for compact hyperkähler manifolds. More precisely, if M is a compact $4m$-dimensional hyperkähler manifold, then 24 divides $m \cdot \chi(M)$ (see [21]).

Appendix A. The Chern character and the Todd class

In this short appendix we will look more closely at the Chern character and the Todd class and how these can be expressed as power series in the Chern classes of the bundle.

Let $p\colon E \to X$ be an n-dimensional complex vector bundle. It follows from the definition of the total Chern class given in Section 3 that we may formally write $c(E) = \prod_{j=1}^{n}(1 + y_j)$ for some 2-dimensional cohomology classes y_j. Then the k-th Chern class $c_k(E)$ is precisely the k-th elementary symmetric polynomial in the y_j's. Furthermore, the Chern character $\mathrm{ch}(E)$ was defined as follows

$$\mathrm{ch}(E) = \sum_{j=0}^{\infty} \frac{1}{j!}(y_1^j + \ldots + y_n^j).$$

But the summands of this power series are symmetric in the y_j's and so, according to the fundamental theorem on symmetric polynomials, they can be uniquely written as polynomials in the elementary symmetric polynomials, i.e. the Chern classes $c_k(E)$.

Let $t_1, t_2, \ldots, t_n$ be indeterminates and $\sigma_1, \sigma_2, \ldots, \sigma_n$ the elementary symmetric polynomials in these indeterminates. The symmetric polynomial s_j, called a *Newton polynomial*, such that $t_1^j + t_2^j + \ldots + t_n^j = s_j(\sigma_1, \sigma_2, \ldots, \sigma_n)$ can be computed recursively by the formula

$$s_j = \sigma_1 s_{j-1} - \sigma_2 s_{j-2} + \ldots + (-1)^{j-2}\sigma_{j-1}s_1 + (-1)^j j\sigma_j.$$

Hence, we have for example

$$s_1 = \sigma_1, \quad s_2 = \sigma_1^2 - \sigma_2, \quad s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3, \quad s_4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2 - 4\sigma_4$$

and the Chern character may be written as

$$\mathrm{ch}(E) = \dim E + c_1(E) + \frac{1}{2}\big(c_1(E)^2 - 2c_2(E)\big) + \frac{1}{6}\big(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)\big) + \ldots$$

Similarly, the Todd class was defined by

$$\mathrm{td}(E) = \prod_{j=1}^{n} \frac{y_j}{1 - e^{-y_j}},$$

which is a power series symmetric in the y_j's. Therefore, it may be again expressed as a power series in the Chern classes of the vector bundle E. However, in this case there is no recursive formula as above and so the summands must be computed directly. If we write $\mathrm{td} = 1 + \mathrm{td}_1(c_1, \ldots, c_n) + \mathrm{td}_2(c_1, \ldots, c_n) + \ldots$, then for example (see [15])

$$\mathrm{td}_1 = \frac{1}{2}c_1, \quad \mathrm{td}_2 = \frac{1}{12}(c_1^2 + c_2), \quad \mathrm{td}_3 = \frac{1}{24}c_1c_2,$$

$$\mathrm{td}_4 = \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4), \quad \mathrm{td}_5 = \frac{1}{1440}(-c_1^3c_2 + c_1^2c_3 + 3c_1c_2^2 - c_1c_4),$$

$$\mathrm{td}_6 = \frac{1}{60480}(2c_1^6 - 12c_1^4c_2 + 5c_1^3c_3 + 11c_1^2c_2^2 - 5c_1^2c_4 + 11c_1c_2c_3 - 2c_1c_5 + {} + 10c_2^3 - 9c_2c_4 - c_3^2 + 2c_6).$$

Let us note finally, that in the thesis we deal mainly with the Todd class of a complexified real vector bundle V. But then we have $c_{2j}(V \otimes \mathbb{C}) = (-1)^j p_j(V)$ from the definition of the Pontryagin classes and, moreover, $c_{2j+1}(V) = 0, j \geq 0$. This is because $V \otimes \mathbb{C}$ is isomorphic to the conjugate bundle $\overline{V \otimes \mathbb{C}}$, whose Chern classes are $c_k(\overline{V \otimes \mathbb{C}}) = (-1)^k c_k(V \otimes \mathbb{C})$. Therefore, $c_{2j+1}(V \otimes \mathbb{C}) = -c_{2j+1}(V \otimes \mathbb{C})$, which implies that over the rationals the odd-dimensional Chern classes are zero.

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