Circular Chromatic Index of the Szekeres snark

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Abstract

One of the refinements of the chromatic index which has been recently investigated is the circular chromatic index. A *circular edge-coloring* of a graph G is a mapping $c\colon E(G) \to [0, r)$ satisfying $1 \leq |c(e) - c(f)| \leq r - 1$ for any adjacent edges e and f. The *circular chromatic index* is the infimum of the set of all r such that there exists a circular r-edge-coloring of the graph G. This invariant is especially interesting for snarks, that is, bridgeless cubic graphs with chromatic index four.

We discuss one of the known infinite families of snarks, namely generalized Szekeres snarks, and we determine the exact value of the circular chromatic index of the two smallest snarks from this family. The smallest graph of order 50 has the circular chromatic index $3 + 2/11$. This solves an open problem posed by Ghebleh (2008). Moreover, we establish a non-trivial upper and lower bound on the circular chromatic index of generalized Szekeres snarks.

Keywords: circular chromatic index, r-coloring, edge-coloring, Szekeres snark

Contents

Introduction

Edge-colorings have played an important role in the history of graph theory. A lot of research has been done on various tasks in this area. However, there are still many problems and questions that are awaiting to be answered, and many new problems have arisen recently.

One of these problems is the determination of the circular chromatic index of graphs. This is especially interesting for snarks, bridgeless cubic graphs that are not 3-edge-colorable.

In the standard 3-edge-coloring of cubic graphs we use three colors usually represented by integers $0, 1, 2$. When we use real numbers instead of integers, we get a circular edge-coloring. This coloring may be viewed as a refinement of the standard edge-coloring.

Time in mathematical models is often represented by real variable. Circular colorings allows us to find optimal solutions to some problems, for example certain variants of scheduling problems.

There are families of snarks for which the circular chromatic index has already been determined: Isaacs' flower snarks (Král et. al., 2006 [13]), Goldberg snarks (Ghebleh, 2007 [8]), generalized Blanusa snarks type 1 (Mazák, 2007 [4]), generalized Blanusa snarks type 2 (Ghebleh, 2008 [9]).

In this work we discuss the circular colorings and the known methods for determining the circular chromatic index. After describing the family of generalized Szekeres snarks, we analyse their structure from the viewpoint of circular colorings. We establish the upper and the lower bound on the circular chromatic index of the whole family that are dependent of the order. Moreover, we present the exact values of the circular chromatic index of the smallest snarks from this family.

Chapter 1

Circular Colorings

1.1 Definitions

A circular edge-coloring is a generalization of the standard edge-coloring. Before we deal with circular edge-colorings, we should define the standard ones.

Definition 1.1.1. An *r-edge-coloring* of a graph G is a mapping $c\colon E(G) \to \{0, 1, 2, \dots, r-1\}$ satisfying

$$1 \leq |c(e) - c(f)| \leq r - 1$$

for any adjacent edges e and f of a graph G.

The smallest integer r for which an r-edge-coloring exists, is the *edge-chromatic number*, or the *chromatic index*, of the graph G. It is denoted by $\chi'(G)$. We say that a graph G is *r-edge-colorable* if G has an r-edge-coloring.

This means that no two adjacent edges in an edge-coloring of a graph G receive the same color. We consider only the graphs that are finite and simple in this work, therefore the value of $\chi'(G)$ is always either Δ or $\Delta + 1$ according to the Vizing's theorem. The symbol Δ denotes the maximum degree of a graph.

Given a graph G, its line graph $L(G)$ is a graph in which each vertex represents exactly one edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are adjacent in G.

An edge-coloring of a graph G corresponds to a vertex coloring of the graph L(G). Therefore many properties of vertex-colorings can be easily applied to edge-colorings.

When we use real numbers instead of integers from Definition 1.1.1, we get a circular edge-coloring.

Definition 1.1.2. Let $r \geq 2$. A *circular r-edge-coloring* of a graph G is a mapping $c\colon E(G) \to [0, r)$ satisfying

$$1 \leq |c(e) - c(f)| \leq r - 1$$

for any adjacent edges e and f of a graph G.
The *circular chromatic index* of a graph G is the infimum of the set of all r such that there exists a circular r-edge-coloring of G. It is denoted by $\chi'_c(G)$. We say that a graph G is *circular r-edge-colorable* if G has a circular r-edge-coloring.

There are also other definitions of the circular coloring whose reading might clarify why the term circular is used. We present such a definition taken from [5].

Definition 1.1.3. Let C be a circle of lenght r. A *circular r-edge-coloring* of a graph G is a mapping c which assigns to each vertex x an open unit length arc $c(x)$ of C, such that for every edge (x, y) of G, $c(x) \cap c(y) = \emptyset$.

The equivalency of Definition 1.1.2 and Definition 1.1.3 was proved in [4].

Remark. As long as we discuss only the edge-colorings, we will use the term "coloring" instead of "edge-coloring" in this work.

1.2 (p, q)-coloring of a graph

Why is it better to use real numbers instead of integers? It is easy to see that when we have an r-coloring of a graph G, then for every $r' > r$, an r'-coloring exists for the graph G. Also every r-coloring is at the same time a circular r-coloring—we use only integers in this case.

We can conclude that $\chi'_c(G) \leq \chi'(G)$. An example of the circular chromatic index that is less than the chromatic index is shown in Figure 1.1. Time in mathematical models is often represented by real variable so the circular chromatic index of a graph allows us to find optimal solutions to some problems, for example certain variants of scheduling problems.

Theorem 1.2.1. *For every graph G, $\chi'(G) - 1 < \chi'_c(G) \leq \chi'(G)$.*

Working with real numbers can be a little bit problematic in computer programming. Therefore, we will employ another definition of circular coloring, which uses only integers when coloring a graph. It was introduced by Vince in [10].

Definition 1.2.2. Let p and q be positive integers. A (p, q)-*coloring* of a graph G is a mapping $c\colon E(G) \to \{0, 1, 2, ..., p-1\}$ satisfying

$$q \leq |c(e) - c(f)| \leq p - q$$

for any adjacent edges e and f.

A (p, q)-coloring of a graph G is equivalent to a (p/q)-coloring of G [5]. It is known that the circular chromatic index is always attained for a finite graph and its value is rational [5]. Hence, we can say

$$\chi'_c(G) = \min\left\{\frac{p}{q} : \text{there exists a } (p, q)\text{-coloring of } G\right\}.$$

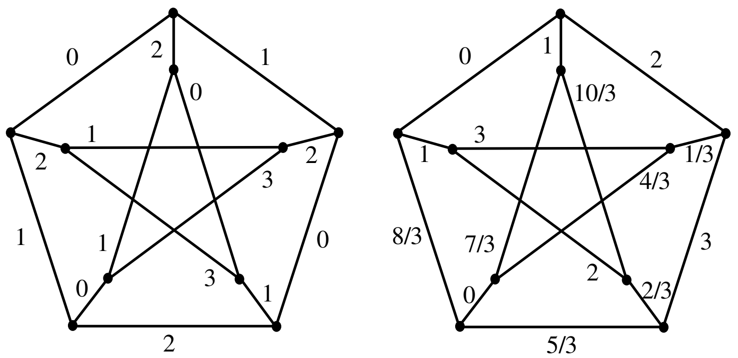

Figure 1.1: The Petersen graph.
$\chi'(G) = 4$ and $\chi'_c(G) = 11/3$.

As it was said before, when we set $q = 1$ we get the standard edge-coloring of G. This proves that the circular coloring is indeed a generalization of the standard coloring.

Chapter 2

The Circular Chromatic Index of a Snark

2.1 Snarks

Snarks are a fundamental class from the viewpoint of colorings and flows. The four colour theorem is equivalent to a statement that no snark is planar. A research on a circular coloring of snarks can bring a new and deeper look at the structure of snarks which is important because of the conjectures that has been opened for years (Cycle Double Cover Conjecture, Tutte's 5-flow conjecture).

Definition 2.1.1. A *snark* is a bridgeless cubic graph with chromatic index four.

It means that the edges of a snark cannot be properly colored by three colors. Using the circular coloring, we can determine how "close to three" the chromatic index of a snark is.

According to Theorem 1.2.1,

$$3 < \chi'_c(G) \leq 4.$$

However, not every fraction from the interval $(3, 4]$ is actually the circular chromatic index of some snark. It has been proved in [11] that the circular chromatic index of a snark is at most $11/3$, which is the index of the Petersen graph (Figure 1.1 and Figure 2.1).

So there is a gap $(11/3, 4)$ in the set $S = \{\chi'_c(G)\colon\ G \text{ is a cubic graph}\}$. It has been conjectured that $S = \{3 + 2/k, k \geq 2\}$. We don't know the graphs with the circular chromatic index of some particular values from this set. Such value is for example $3 + 2/11$. This problem has been posed by

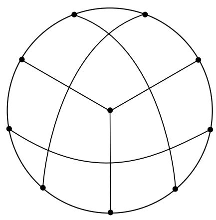

Figure 2.1: The Petersen graph.

Ghebleh in 2008. On the contrary, it has been proved by Mazák in [4] that all numbers of the form $3 + 2/3k$ belong to the set S.

Many researchers believe that the circular chromatic index of a bridgeless cubic graph other than the Petersen graph is at most $7/2$ and $(7/2, 11/3)$ is another gap in S.

2.2 Determining the circular chromatic index

For a given graph G and a given number r deciding whether $\chi'_c(G) < r$ is NP-complete. In this section we will discuss the most common techniques for determining the circular chromatic index.

Checking all possible colorings

For graphs with small number of vertices a backtracking algorithm on a computer can be used. There is only a finite number of fractions which could be the circular chromatic index of a given graph.

A lot of snarks can be divided into smaller blocks (mostly one small block is used repeatedly) on which a backtracking algorithm is run. An example of such blocks is shown in Figure 3.2. As a result of this computation, we get all the possible colorings of the *semiedges* that is edges incident with only one vertex. If there exists a way to match the colored blocks together, we will get the desired coloring for the whole graph.

Tight cycles

Another useful tool for determining the circular chromatic index is the concept of tight cycle [7].

Definition 2.2.1. Let c be a (p, q)-vertex-coloring of a graph G. Let $u_0u_1...u_{n-1}$ be a cycle in G, $u_i \in V(G)$ and $u_n = u_0$. This cycle is called a *tight cycle* if and only if

$$c(u_{i+1}) = (c(u_i) + q) \textit{ modulo } p$$

for every $i \in \{0, 1, ..., n-1\}$.

When we consider an edge-coloring of G, we look for a tight cycle in its line graph $L(G)$. A coloring of edges in a graph G is equivalent to a coloring of vertices in $L(G)$.

Why is finding a tight cycle so useful? The following lemmas which are an adaptation of the lemmas in [7] show how to apply this approach to determine the circular chromatic index.

Lemma 2.2.2. *$\chi'_c(G) = r$ if and only if G has a circular r-edge-colouring and every circular r-colouring of $L(G)$ has a tight cycle.*

Lemma 2.2.3. *If $\chi'_c(G) = p/q$, where p and q are coprime integers, then $L(G)$ has a cycle C of length kp for some positive integer k. Every color from $\{0, 1, \ldots p-1\}$ is used in every (p, q)-coloring of G.*

We can precolor some cycle in $L(G)$ as a tight cycle and try to color the other vertices. The bigger the nominator p is, the more vertices are colored. This can also give us some other restriction on how large the fractions p/q can be.

It is easy to see from the above lemma that in a (p, q)-coloring of a graph G with $\chi'_c(G) = p/q$ every color from $\{0, 1, \ldots, p-1\}$ has to be used at least once.

$(3 + \varepsilon)$-coloring

This technique is useful for establishing the lower bound of the circular chromatic index. We have used it to establish a bound on the circular chromatic index of generalized Szekeres snark.

Instead of integers (or fractions) we assign to each edge of a graph an r-circular interval [4, 8, 9]. The interval represents a set of possible colors for the edge. The following definition was taken from [8].

Definition 2.2.4. For any $a, b \in [0, r)$, the *r-circular interval* $[a, b]_r$ is defined as follows:

$$[a, b]_r = \begin{cases} [a, b] & (a \leq b) \\ [a, r) \cup [0, b] & (a > b) \end{cases}$$

If a or b are out of the interval $[0, r)$ we usually reduce them modulo r. Every real number has a representative in $[0, r)$ when reduced modulo r. If $b - a \geq r$ then $[a, b]_r = [0, r)$.

In our colorings we will set $r = 3 + \varepsilon$. We consider three edges incident with one vertex in a cubic graph. The intervals are assigned to the edges as follows. When we “color” one edge with interval a, the other two have the colors from the intervals $a + [1, 1 + \varepsilon], a + [2, 2 + \varepsilon]$.

Chapter 3

Generalized Szekeres Snark

The Szekeres snark was discovered in 1973 [2]. It is composed of five copies of the Petersen graph with some edges cut into semiedges and joined together as shown in Figure 3.1.

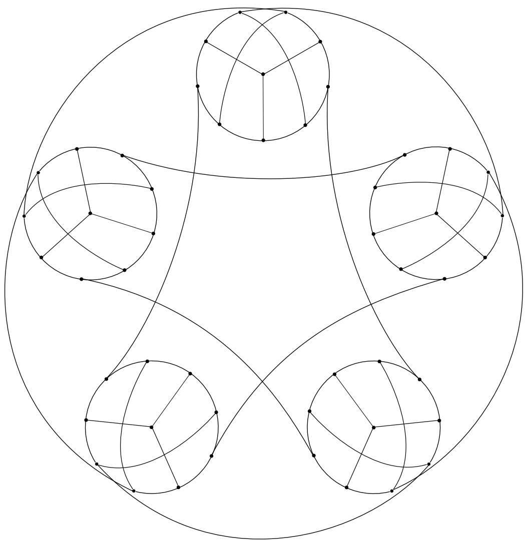

Figure 3.1: The Szekeres snark.

3.1 Generalized Szekeres snarks

An infinite family of snarks that contains the Szekeres snark was discovered by Watkins [2].

In the construction of generalized Szekeres snarks we use the block A (see Figure 3.2) with four semiedges a, b, c, d. When we join the semiedge a to

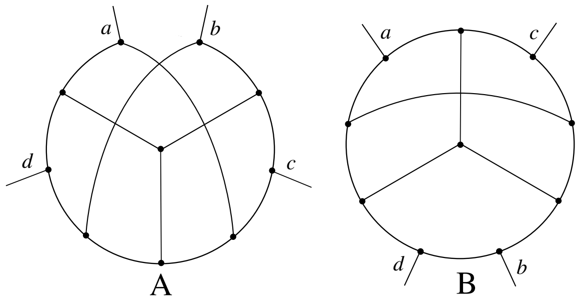

Figure 3.2:

the semiedge b and the semiedge c to d, we get the Petersen graph. We will call this block the Petersen block.

The generalized Szekeres snark of order 50, which is the smallest snark from this family, contains five copies of the Petersen block. The way they are joined together is shown in Figure 3.3. Generalized Szekeres snark of order 90 is shown in Figure 3.4. (We use the square instead of the Petersen block).

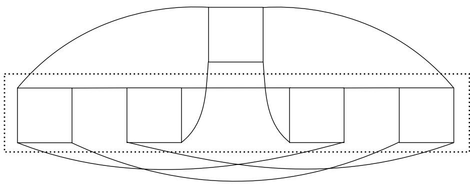

Figure 3.3: The generalized Szekeres snark of order 50.

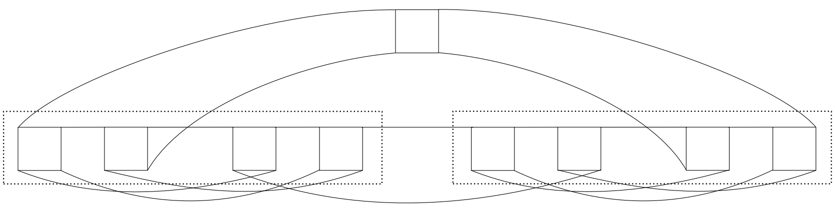

Figure 3.4: Generalized Szekeres snark of the order 90.

To get an infinite family of snarks we just keep adding another groups of four Petersen blocks to the graph we have already constructed. One such group is shown in Figure 3.3 marked by a frame. We will call the block to which the *four-blocks* are connected *the main Petersen block*. Individual snarks from this family have orders $50, 90, 130, \ldots$ (Figure 3.5). We denote the graph containing m four-blocks by S_m.

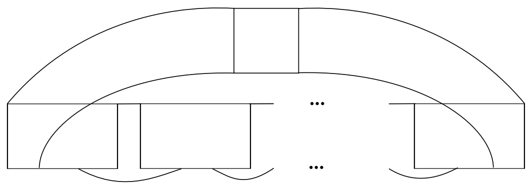

Figure 3.5: Generalized Szekeres snarks of orders $50, 90, 130, \ldots$.

We can also get another infinite family of snarks, when we use the block B instead of the block A. The orders of the snarks we obtain are similarly $50, 90, 130, \ldots$ (Figure 3.6).

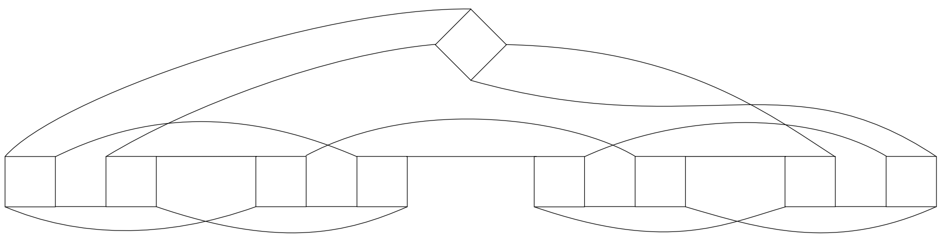

Figure 3.6: The snark of order 90 constructed from the block B.

Another method how to construct snarks is to use blocks A and blocks B mixed together.

3.2 Building blocks and 3-colorability

We will show that a generalized Szekeres snark is indeed a snark in this section. We try to color the blocks that are used in the construction of this snark by three colors.

3-coloring of the Petersen block

There is only a limited number of 3-colorings of the Petersen block. They are shown in Figure 3.9 (other 3-colorings can be obtained from these by a permutation of the colors). We will show that there exist no other 3-coloring of the Petersen block A.

The Petersen block has four semiedges a, b, c, d. We show that $c(a) \neq c(b)$ for any 3-coloring of A.

In Figure 3.7 we have colored without loss of generality the semiedge a and its two adjacent edges by colors $0, 1, 2$. The semiedge b has the color 0. We have four edges adjacent with the four edges that are already colored, so there are four possible colorings of the block. When we try to color the remaining edges we get a contradiction—two adjacent edges with the same color (marked by a circle).

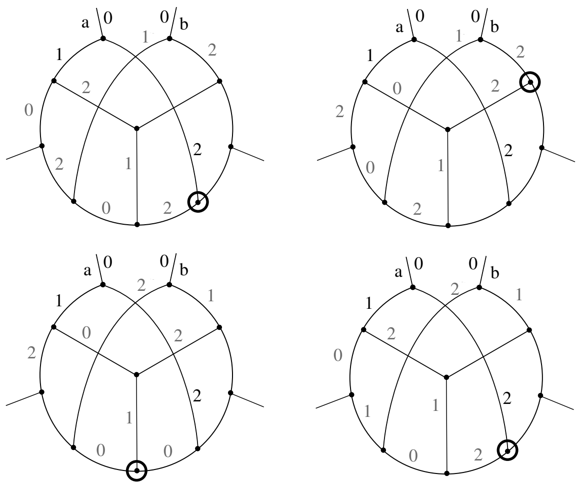

Figure 3.7: $c(a) = c(b)$

The semiedges c and d cannot either be colored by the same color, because the Petersen graph is edge-transitive. There exists an automorphism which maps the original edge ab to the original edge cd.

Moreover, the semiedges a, b, c, d cannot be colored by 3 different colors. Without loss of generality let $0, 1, 2$ be the colors of a, b, c. All possible results for this precoloring are shown in Figure 3.10. The graph is symmetric so it does not matter if we color c or d.

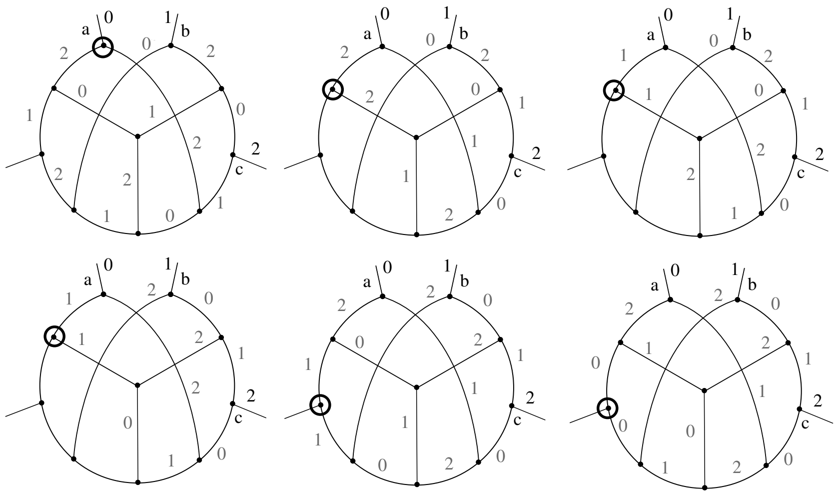

Figure 3.8: a, b, c are of different colors

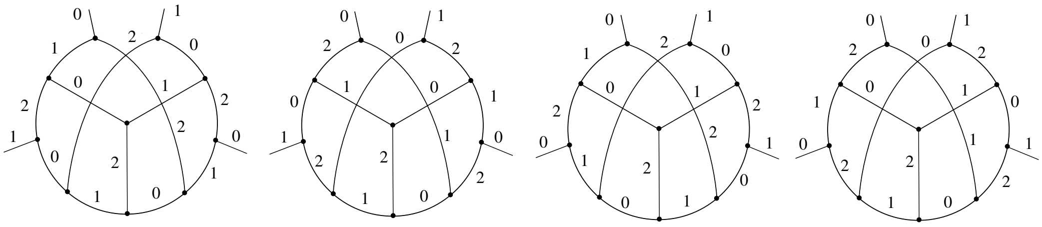

Figure 3.9: All possible 3-colorings.

Therefore, the colors of the semiedges a and b are different. The color of the semiedges c and d is different. We cannot use 3 colors for all semiedges. That means that both colors $c(a)$ and $c(b)$ are also used as colors of c and d, but not necessarily in this order.

3-coloring of a four-block

To make it simple and more transparent we use the knowledge of the 3-colorings from the previous section. So we color only the edges that are semiedges in Petersen blocks. (We will still call them semiedges even if there are no semiedges in a four-block.)
We will use the following notation: $A.a$ will denote the semiedge a of the block A, $c(A.a)$ will denote the color of this semiedge.

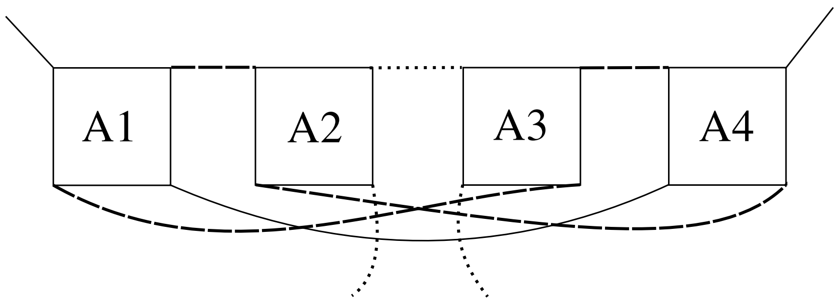

Figure 3.10: A 3-coloring.

We color the semiedges of the first block A_1 so that $c(A_1.a) = c(A_1.c)$. It is shown in Figure 3.10 where we use different types of lines instead of colors. The colors of the other edges are determined by this initial coloring. $A_3.c$ is dashed, therefore $A_3.a$ or $A_3.b$ has to be dashed. As $A_2.b$ cannot satisfy this condition we assign a dashed line to $A_3.b$. We can continue this way until all semiedges are colored. The dotted edges could be also simple lines but the important thing is that $c(A_1.a) = c(A_4.b)$ and $c(A_2.c) = c(A_3.d)$.

Similarly, we color the first block so that $c(A_1.a) = c(A_1.d)$. If $A_2.b$ and $A_2.d$ are dotted, the block A_4 cannot be 3-colored (Figure 3.11). So

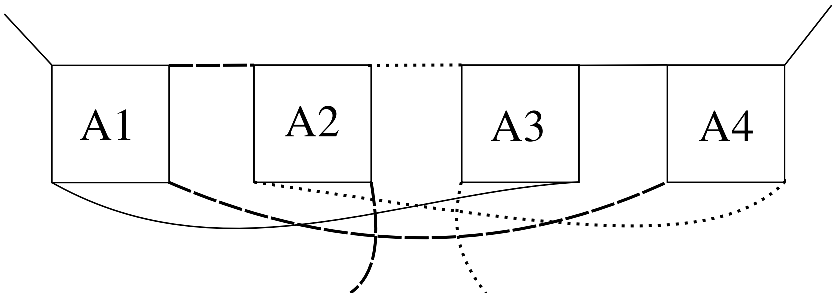

Figure 3.11: An infeasible 3-coloring.

they have to be simple (Figure 3.12). And again $c(A_1.a) = c(A_4.b)$ and $c(A_2.c) = c(A_3.d)$. We analysed all possible 3-colorings of the graph, because the first block cannot be colored in any different way.

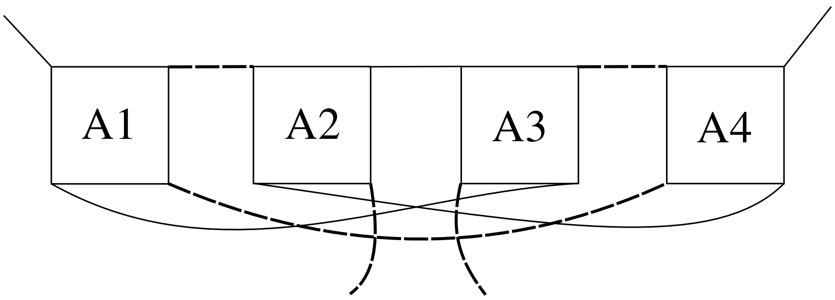

Figure 3.12: A 3-coloring.

Thus for every 3-coloring of this graph, $c(A_1.a) = c(A_4.b)$ and $c(A_2.c) = c(A_3.d)$. We can say that if some color comes into this block by the semiedge $A_1.a$, the same color also leaves by the semiedge $A_4.b$ (we will call this the upper line). The same can be said about the semiedges $A_2.c$ and $A_3.d$ (we will call this the lower line).

The resulting graph, a generalized Szekeres snark, cannot be colored by three colors. No matter how many four-blocks we join together, the color does not change neither on the upper nor on the lower line. That is in a contradiction with the coloring of the main Petersen block, to which the four-blocks are connected, where the semiedges a, b and c, d has to be different.

Chapter 4

Circular Chromatic Index of Generalized Szekeres Snarks

4.1 Circular chromatic index of the smallest snark S_1

We know that $\chi'_c(S_1) \in (3, 4]$, $\chi'_c(S_1) \leq 11/3$ and $\chi'_c(S_1)$ is rational. Therefore, we are looking for a circular chromatic index of the form p/q, where p and q are coprime integers.

It is easy to see that $p \leq |E(G)|$, because we have to use every color from $\{0, 1, \ldots, p-1\}$ at least once (see tight cycles in Chapter 2). Since $p/q > 3$, we have $q < p/3 \leq |E(G)|/3 = |V(G)|/2$.

Theorem 4.1.1. *The circular chromatic index of the Szekeres snark S_1 is $35/11 = 3 + 2/11$.*

Proof. The snark S_1 has 50 vertices. Therefore we are interested only in fractions such that $p \leq 75$ and $q \leq 25$. There is only a finite number of fractions p/q which satisfy this conditions. For this snark some of the fractions that come into account are $11/3 > \cdots > 7/2 > \cdots > 10/3 > \cdots > 13/4 > \cdots > 16/5 > 67/21 > 51/16 > 35/11 > 54/17 > 73/23 > 19/6 > \cdots > 3$.

When we find a (p, q)-coloring, then we have to check only the fractions that are smaller than p/q. For example, we find a $(16, 5)$-coloring but a $(19/6)$-coloring does not exist. Then we have to try only the fractions that are between $16/5$ and $19/6$. It is better to start with the fractions with small p and q because the checking of the existence of such a coloring takes less time. We can exclude quite a lot of fractions in this way.

We have used a backtracking algorithm to find a (p, q)-coloring of the Szekeres snark S_1. However, backtrack on a graph with 50 vertices is unmanageable for large p and q. So we have run a backtrack only on the small block the graph consists of—the Petersen block with 10 vertices (Figure 3.2, the block A). The aim of this backtrack was to acquire all possible (p, q)-colorings of the block.

Having these possible (p, q)-colorings, we could run another backtracking algorithm on the set of all possible colors of the semiedges of the Petersen block. The aim was to find the colors of the semiedges for every Petersen block such that all corresponding semiedges in the snark have the same color. This was not possible in the case there has not existed a (p, q)-coloring for the whole graph.

To make the first faze of this algorithm, the coloring of the Petersen block, more effective we have used the knowledge of the $(3+\varepsilon)$-coloring of the Petersen block (section 2.2) where an interval is assigned to every semiedge of the block. This interval represents a set of possible colors for each semiedge. The total number of colorings of the semiedges for one Petersen block is: $|a\text{-interval}| \times |b\text{-interval}| \times |c\text{-interval}| \times |d\text{-interval}|$ where $|a\text{-interval}|$ is the number of colors from the interval that belongs to the semiedge a.

We have precolored the four semiedges for each coloring and then we have run the backtracking algorithm. Such computation has run faster even if we have to run the algorithm more times on one Petersen block.

The optimal coloring of the snark S_1 is shown in Figure 4.1.

□

Theorem 4.1.2. *The circular chromatic index of the snark S_2 is $25/8 = 3 + 1/8$.*

Proof. This result has been obtained by the same method as we have used for the the snark S_1. □

The generalized Szekeres snark S_3 satisfy $\chi'_c(S_3) \leq 71/23 = 3 + 2/23$. There is only one smaller fraction that could possibly be the circular chromatic index of the snark S_3, it is $179/58 = 3 + 5/58$. Other fractions have been excluded by the procedure that has been described before. We conjecture that $3 + 2/23$ is indeed the circular chromatic index of S_3. An approach to prove this result is to find a tight cycle in the graph $L(S_3)$ for every $(71, 23)$-coloring of S_3. The number of these colorings is too big to check all of them in a short time. So far, every $(71, 23)$-coloring of S_3 that has been checked contained a tight cycle.

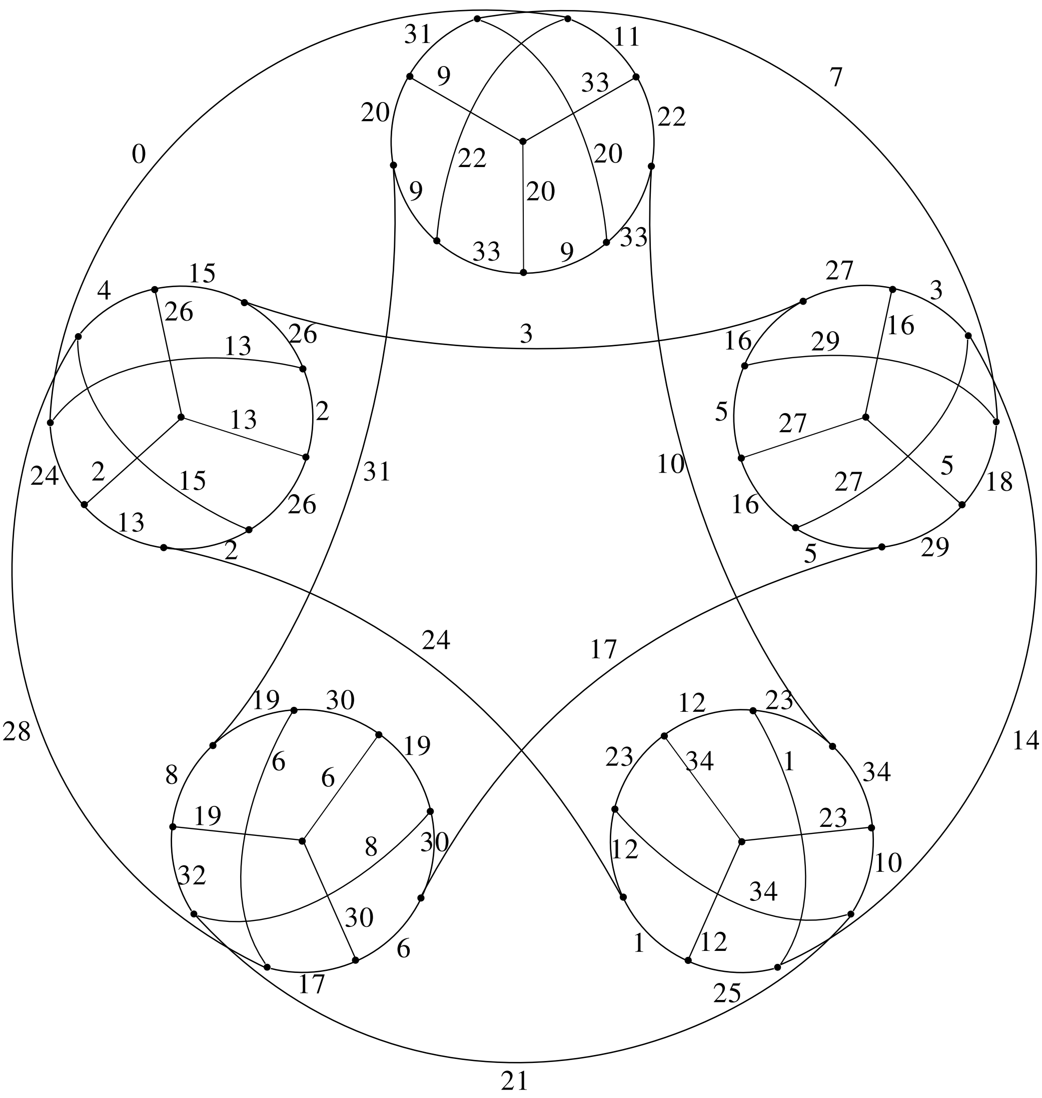

Figure 4.1: A (35,11)-coloring of the Szekeres snark.

The generalized Szekeres snark S_4 satisfy $\chi'_c(S_4) \leq 46/15 = 3+1/15$. The possible values of the circular chromatic index are $3+1/15, 3+5/76, 3+4/61$. Other fractions has been excluded.

4.2 The lower bound

The lower bound on the circular chromatic index of generalized Szekeres snarks is a function of their order. This function has to be at least $3+1/f(n)$, where $f(n)$ is a linear function of $|V(S_m)|$, because the denominator of the chromatic index has to be at most $|V(S_m)|/2$. So $\chi'_c(S_m) \geq 3 + 1/q$, where q is maximal possible.

Theorem 4.2.1. *The circular chromatic index of S_m satisfies*

$$\chi'_c(S_m) > 3 + \frac{2}{n} = 3 + \frac{1}{20m+5}$$

where $n = V(S_m) = 40m + 10$.

This bound is trivial as it reflects one of the basic restrictions for the circular chromatic index. It can be improved by using the knowledge of the $3+\varepsilon$-coloring, where we examine the possible change of colors in the four-block. The bigger the change is, the smaller the circular chromatic index can be.

It can be showed by exhausting case analysis that the change of the color on the upper line is at most 4ε. The same holds also for the lower line. Similarly, it can be showed that when we have a change of 4ε on the upper line, the change on the lower line is at most 3ε. So we cannot have the change of 4ε for each line at the same time in the four-block. We also believe that when we have a change of 4ε on the lower line, the change on the upper line is at most 3ε.

Conjecture 4.2.2. *Let $m > 2$. The circular chromatic index of S_m satisfies*

$$\chi'_c(S_m) > 3 + \frac{1}{4m}.$$

If there exist a $(3 + 1/4m)$-coloring of generalized Szekeres snarks, the change of the color should be big on the upper and on the lower line of the four-block the generalized Szekeres snarks consist of. In this case, we have to obtain the total change of $4m - 1$ on m four-blocks. The main Petersen block cannot be $3 + 1/4m$-colored to acquire less change. Therefore, there must be at least one four-block with the change 4 on the upper and with the change 4 on the lower line. According to what we have said before, such change cannot be attained.

The main Petersen block also allows a change of the color. For m that is small enough this change is essential. Therefore, there is no need for the change on the four-blocks to be so big.

4.3 The upper bound

The circular chromatic index of generalized Szekeres snarks can be arbitrarily close to 3. Bigger graphs of this family have smaller index. Therefore, the upper bound has to be a function of the order of a snark, if we don't want the bound to be trivial.

Theorem 4.3.1. *The circular chromatic index of S_m satisfies*

$$\chi'_c(S_m) \leq 3 + \frac{2}{7m+2}.$$

Proof. Recall the construction of the generalized Szekeres snarks described in Section 3.1. We start by coloring the main Petersen block. According to the Definition 1.2.2 the adjacent edges has to be at distance at least $7m+2$ and at most $14m+6$ in a $(21m+8, 7m+2)$-coloring. This can be easily checked in Figure 4.2.

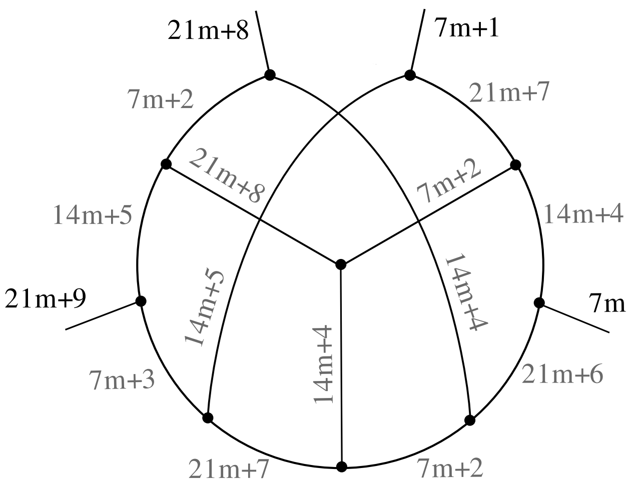

Figure 4.2: A $(21m+8, 7m+2)$-coloring of the Petersen block.

Now we color the four-block. We have to use two different colorings of this block in order to obtain a $(21m+8, 7m+2)$-coloring of the whole graph.

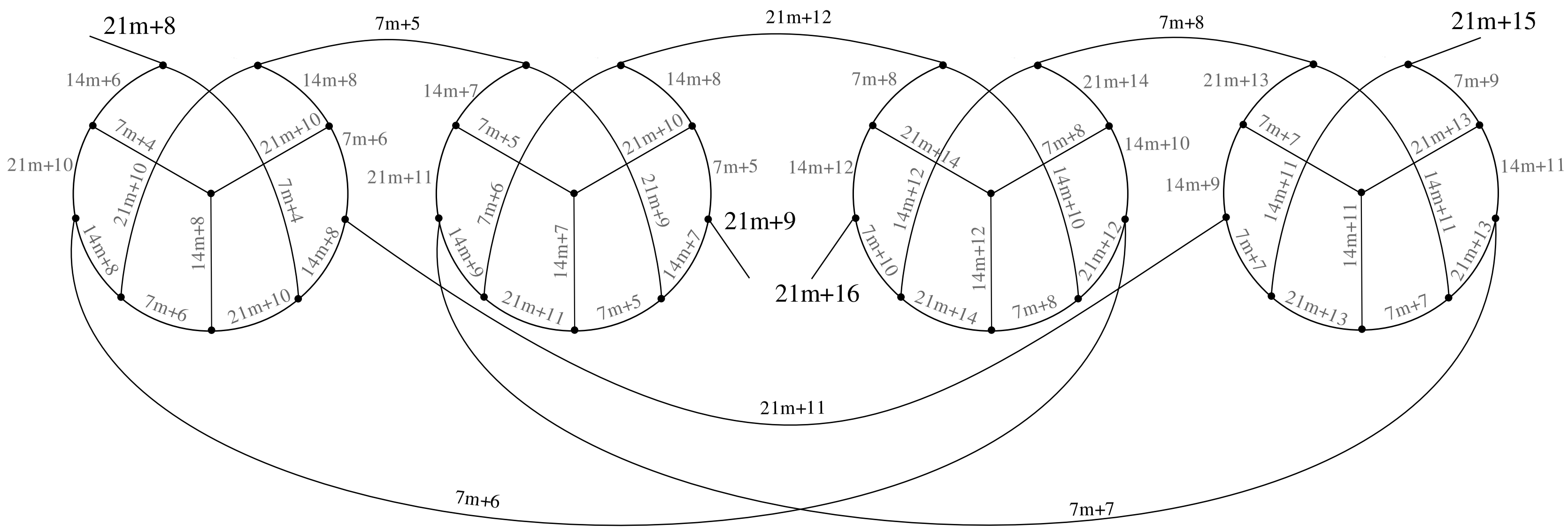

Figure 4.3: A $(21m+8, 7m+2)$-coloring of the four-block—coloring c_1.
The colors $21m+8$ and $21m+9$ increase both by 7.

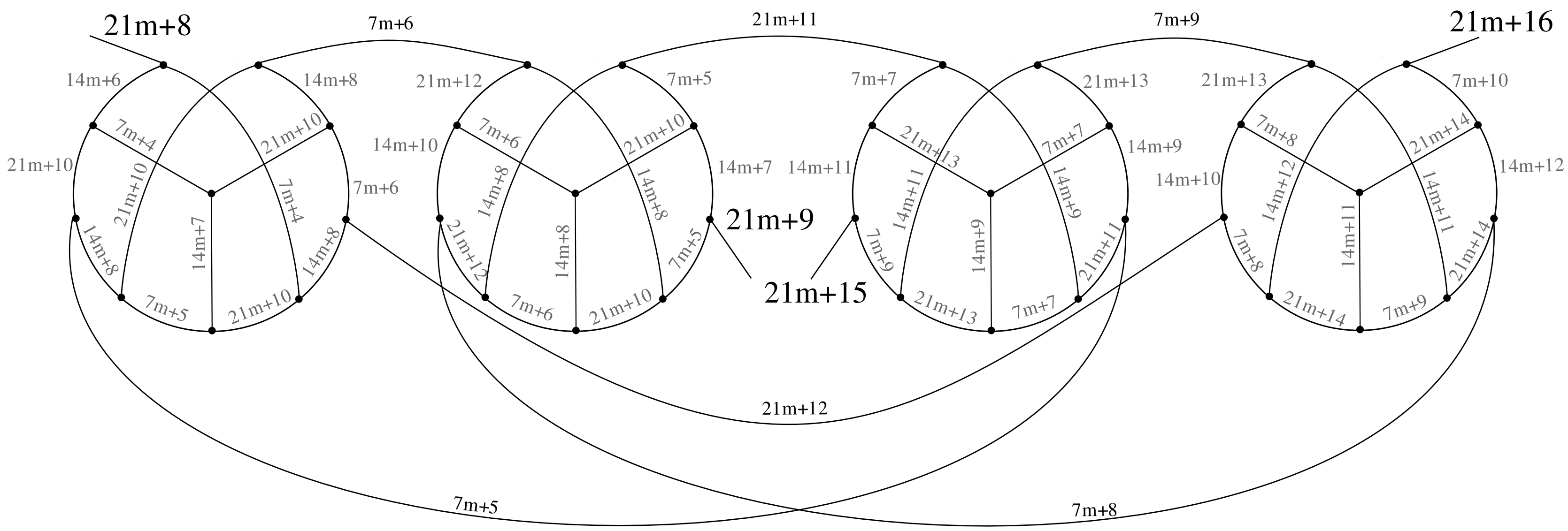

Figure 4.4: A $(21m+8, 7m+2)$-coloring of the four-block—coloring c_2.
The colors $21m+8$ increases by 8 and $21m+9$ increases by 6.

This colorings may look rather complicated, but the reader should pay attention to the semiedges through which the blocks will be joined together. In Figure 4.3 is shown the first coloring. The colors which comes into this block are $0 = 21m + 8$ and $1 = 21m + 9$. Both colors increase by 7 in one block. After m blocks colored analogously we get the change of $7m$ for both lines.

In the second coloring (Figure 4.4) the colors which comes into this block are again $0 = 21m + 8$ and $1 = 21m + 9$. However, the upper line increases by 8 and the lower by 6.

To get the final coloring, we join $m-1$ blocks with the coloring c_1 and one block with the coloring c_2. The colors of the semiedges are 0, $0+7(m-1)+8 = 7m+1$, 1, $1+7(m-1)+6 = 7m$. So they can be joined to the main Petersen block. □

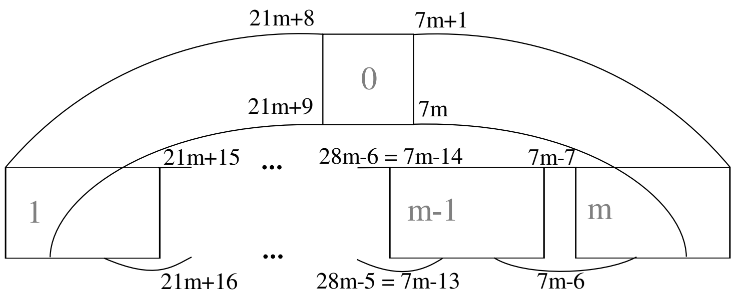

Figure 4.5: A $(21m + 8, 7m + 2)$-coloring of generalized Szekeres snarks.

Note that this colouring is not optimal for the smallest graph in the family of generalized Szekeres snark. However, for the snark S_2 the bound is tight. $\chi'_c(S_2) = 3 + 2/(7 \cdot 2 + 2) = 25/8$.

The upper bound is $3+O(1/m)$, the lower bound is $3 + \Omega(1/m)$, so they are asymptotically the same. The circular chromatic index is $3 + \Theta(1/m)$. We believe that the circular chromatic index of S_m is closer to (or even the same as) the upper bound.

Conclusion

The upper bound that has been established in this work proves that the circular chromatic index of generalized Szekers snarks takes infinitely many values and can be arbitrarily close to three. This family of snarks is therefore another class with this property right after the class of generalized Blanusa snarks which is known to be the first such class of non-trivial snarks.

The circular chromatic index of the smallest snark in the family of generalized Szekeres snarks is $35/11 = 3 + 2/11$. This snark is interesting because it is the first known snark with this circular chromatic index. Therefore, we have solved a problem posed by Ghebleh in 2008.

A plan for further research is to determine the exact value of the circular chromatic index of the generalized Szekeres snarks. The method used for the generalized Szekeres snarks in this work can also be used for the family of snarks constructed from the block B. It might be possible to alter this method and use it for the snarks with blocks A and blocks B mixed together.

After solving the problem, we can formulate similar one. There is no known non-trivial snark with the circular chromatic index $3 + 2/13$ (to the author's knowledge).

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