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σ-VALUATION OF TREES

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Introduction

More than 45 years ago, Ringel conjectured that a complete graph K_{2m+1} can be decomposed into $2m+1$ copies of arbitrary tree T with m edges. Despite a lot of effort, there are still just a few known tree classes for which this conjecture is proven.

Main object of this paper is to show that it is possible to find such decomposition for arbitrary spider. We are going to prove this by showing that every spider has a σ-Valuation. Besides that, we will extend this result to a tree class, obtained by substituting paths in spiders by graphs from a subclass of caterpillars. We will also show that there exists σ-valuation of some other graph classes.

In first part of this paper, we will introduce some basic concepts, definitions and constructions known in a field of graceful labelings. We will also mention certain classes of trees for which it is proven that they could be labeled by some labeling, connected with Ringel's conjecture.

Second part consist of our main results. σ-valuation of spiders is proven in two steps. The proof is constructive, so it is possible to use it for finding such labeling for any spider. We define in this chapter superclass of spiders which also admits σ-valuation.

In the last part we show how to use graceful labelings of graphs for constructing σ-valuations of certain larger classes. We use such construction to show that every lobster, irregular banana and irregular bamboo tree has a σ-valuation. Besides that we also show that every tree with diameter at most 7 and every tree with at most 38 vertices have σ-valuation.

KEYWORDS: Ringel's conjecture, σ-valuation, ρ-valuation, Graceful tree labeling, Lobster, Spider

Chapter 1

Definitions and known results

Conjecture 1.1. *(Ringel'63) Let K_{2m+1} be complete graph with $2m+1$ edges and T an arbitrary tree with m edges. Then K_{2m+1} can be decomposed into $2m+1$ copies of T.*

This conjecture is still unsolved. Rosa introduced in 1966 valuations of trees which could lead toward a proof of the Ringel's conjecture. We will first introduce the most well known of this labelings – the Graceful labeling.

Definition 1.1. Graceful labeling *of a graph $G = (V, E)$ is a labeling $f : V \to \{0 \dots |E|\}$ inducing an edge labeling g, defined by $g(uv) = |f(u) - f(v)|$, such that:*

- $\forall u, v \in V, u \neq v, f(u) \neq f(v)$
- *g is a bijection from $E(G)$ to $\{1 \dots |E|\}$*

If graph G has a graceful labeling, we say that G itself is graceful. Graceful labeling is also called *β-valuation*.

Conjecture 1.2. *(Graceful labeling conjecture) Every tree has a graceful labeling.*

It is easy to see that if labeling f is graceful then also the labeling $h(v) = |E| - f(v)$ is graceful. Such labeling is called the *complementary labeling*. First condition trivially holds. For each edge $f(uv) = |u - v| = |v - u| = ||E| - |E| + v - u| = |(|E| - u) - (|E| - v)| = h(uv)$, so the complementary labeling also preserves the second condition. This implies that every nontrivial graph has an even number of graceful labelings.

Rosa introduced an even more restrictive labeling – bipartite. There is no conjecture analogous to GLC for this valuation, because there are trees for which it is proven that they cannot be bipartitely labeled. In spite of this fact, is bipartite labeling still very interesting, because of some it's interesting properties.

Definition 1.2. *The* Bipartite labeling *is graceful a labeling with a further property that there exists $x \in \{0 \dots |E|\}$ such that for an arbitrary edge uv either $f(u) \leq x < f(v)$ or $f(v) \leq x < f(u)$ holds.*

A tree which admits bipartite labeling is also called *balanced*, *interlaced* or *α-valuation*.

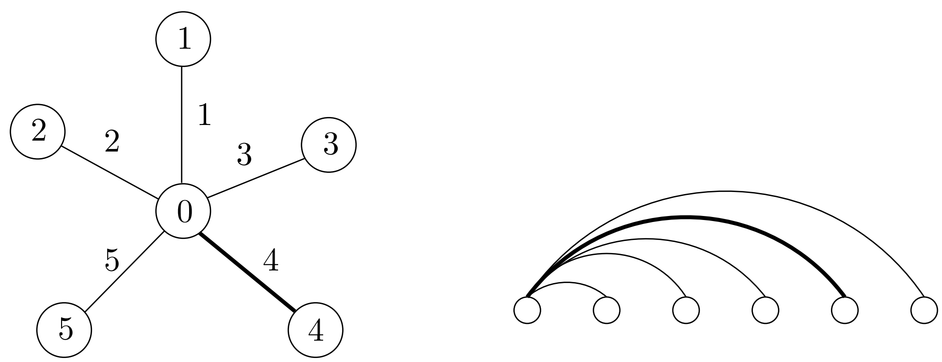

Figure 1.1: Two possible representations of an graceful labeling.

Two various representations of one specific labeling of a star can be seen on figure 1.1. On the left representation are the values of edge and vertex labels written on respective places. On the right we could see representation, where axis represents possible values of vertex labels and every vertex is placed on the plane according to its label value. We could then count the value of an edge, by just looking at the distance between embedded vertices. Highlighted line in both representations corresponds to the same edge.

The bipartite labeling allows us to do some operations to generate new graceful labelings.

Definition 1.3. *Let f be a bipartite labeling of a graph G with bipartition (A, B) with k such that there exists such vertex v that $v \in B, f(v) = k$ and*

$u \in A, f(u) = k-1$. *We call* f' *the* reverse *labeling of* f *where* f' *is defined by:*

$$f'(x) = \begin{cases} k-1-f(v), v \in A \\ |E|+k-f(v), v \in B \end{cases}$$

If f is a bipartite labeling then also f' is. For any bipartite labeling f, we can by using reverse and complementary labelings, label with zero such vertices for whose $f(v)$ has one of values $0, k-1, k, |E|$

There are two more kinds of near graceful labelings whose do not fulfill conditions of gracefulness, but could still lead to proof of Ringel's conjecture.

Definition 1.4. *The* σ-valuation *is a labeling, where the induced edge labeling must satisfy the condition of being bijection from* $E(G)$ *to* $\{1 \dots |E|\}$*, but the vertex range is relaxed to* $\{0, \dots, 2|E|\}$

Definition 1.5. *The* ρ-valuation *is a labeling, where the induced edge labeling is relaxed to* $\{1, \dots, 2|E|\}$*, under the condition that either label* i *or label* $2|E|+1-i$ *is used, but not both and the vertex range is relaxed to* $\{0, \dots, 2|E|\}$.

It could be seen that there is a hierarchy of labelings. Sorted from strongest to weakest it is α, β, σ and ρ.

We will now define some labelings that could be interesting as building blocks for various valuations.

Slater and Maheo with Thullier independently introduced the *k-graceful* labeling scheme

Definition 1.6. *The* k-graceful *labeling* f *is a labeling such that*

- f *is an injection into* $\{0, 1, \dots, |E|+k-1\}$
- *edge labels are shifted to* $\{k, \dots, |E|+k-1\}$

Every graph with a bipartite graceful labeling is k-graceful for all k, since we can add arbitrary constant to the labels from partition with higher label values. But there are graphs which are k-graceful for all k and do not have a bipartite graceful labeling.

Definition 1.7. *The* (k,d)-graceful *labeling* f *is a labeling such that*

- *f vertex labels are in range* $\{0, 1, \ldots, (|E|-1)d + k\}$
- *edge labels take values* $\{k, k+d, k+2d, \ldots, (|E|-1)d\}$

Theorem 1.1.
- *Every k-graceful graph is (kd, d)-graceful.*
- *Every connected (kd, d)-graceful graph is k-graceful.*
- *If graph is (k, d)-graceful and not bipartite then $k \leq (m-2)d$*

Definition 1.8. *The* caterpillar tree T *is a tree which consists of a path P_n and a set of vertices not on P_n, each of them joined to exactly one vertex on P_n.*

Theorem 1.2. *Every caterpillar is graceful.*

Proof. (due to Rosa [18]) There is an algorithm for labeling which could be used for every caterpillar. To label a path – a special case of caterpillars – one could label the vertices on this path by alternatively using largest and smallest label possible. Similar approach is applicable for any caterpillar. One has to take a vertex which is a beginning of a longest path in the tree and start labeling vertices along this path. For every vertex on this path one has to use largest(smallest) label possible then label all adjacent vertices not lying on that path with consecutive available labels of opposite size and move to next vertex, which would be labeled with smallest(largest) label possible. □

Maheo in 1980 defined that a graph is *strongly graceful*, if it possesses a bipartite graceful labeling with the extra condition that for every vertex v the labels of all incident edges form a sequence of consecutive integers. As we have seen, labeling from the proof of the last theorem is strongly graceful, so every caterpillar is strongly graceful. Bodendiek and Schumacher have shown that caterpillars are the only trees with this property.

It has to be mentioned that today is therm strongly graceful labeling denoting also other labeling, introduced by Broersma and Hoede. In this paper we will use this therm for labeling as it was defined by Maheo.

Definition 1.9. *A* lobster T *is a tree consisting of path P_n and vertices not on P_n at distance at most two form a vertex in P_n.*

Even though we could get a caterpillar from any lobster simply by deleting all its leaves, there is very little known about gracefulness of lobsters, up to some special cases.

Caro et al. [7] proved the following:

Theorem 1.3. *All lobsters have a ρ-labeling.*

Huang, Kotzig and Rosa studied gracefulness of trees with at most 4 leaves and trees not admitting bipartite labeling in 1982 paper [12], we will present here some of their results:

Definition 1.10. *Let T be a tree and v a vertex of T. A* branch vertex *of T is a vertex of degree at least 3 in T. A* v-endpath *of T is a path P from v to a leaf of T such that each internal vertex of P has a degree two in T. A* spider $S(a_1, \ldots, a_r)$ *is a tree with exactly one branch vertex v and v-endpaths of lengths $1 \leq a_1 \leq \ldots \leq a_r$ where $r = degv$. These v-endpaths will be called the* legs.

Theorem 1.4.

- *The tree $S(p, q, r)$ with 3 leaves has a bipartite labeling if and only if $(p, q, r) \neq (2, 2, 2)$*
- *Every tree $S(p, q, r)$ with 3 leaves has a graceful labeling.*

Trees with exactly four leaves can be divided into two groups. First can be described by using the spider notation as $S(p, q, r, s)$. Second group are trees with two branch vertices u and v, both with degree 3. We would describe this trees by using a notation $(p, q; r; s, t)$, where the numbers refer to lengths of the u-endpaths, distance between u and v and lengths of the v-endpaths respectively.

Theorem 1.5.

- *If at least two of p, q, r, s are not equal 2 then there exists a bipartite labeling of the spider $S(p, q, r, s)$.*
- *Every tree $S(p, q, r, s)$ has a graceful labeling.*
- *Every tree $S(p, q; r; s, t)$ has a graceful labeling.*

Corollary 1.1. *All trees with at most 4 leaves are graceful.*

Theorem 1.6. *Let T_{rs} denote a tree of diameter three on $r + s + 2$ vertices with two vertices of degree $s + 1$ and $r + 1$ adjacent to each other and $r + s$ leaves. Let P_{rs} be the tree of diameter six obtained by replacing each edge in T_{rs} with a path of length two. Then next claims holds.*

- *The tree P_{rs} has a bipartite labeling if and only if $|r-s| \leq 1$.*

- *Every tree P_{rs} has a graceful labeling.*

As we have already mentioned, it is shown that not every tree has a bipartite graceful labeling. There are some results describing some such graph classes.

Huang, Kotzig and Rosa proved next theorem:

Theorem 1.7. *Let T be a tree all of whose vertices are of odd degree. Let T^* be obtained from T by replacing every edge of T by a path of length two. If $|V(T)| \equiv 0 (mod\ 4)$ then T^* does not have a bipartite labeling.*

Theorem 1.8. *Let T be a tree with diameter 4 and T is not a caterpillar nor path then T has no bipartite labeling.*

Theorem 1.9. *Let $T(q_1, q_2, q_3; s)$ be a tree rooted at its centre x. Vertex x is adjacent to s leaves and to 3 vertices having degrees $q_i + 1$, where each of these are adjacent only to x and the leaves of T. In this case, the following holds:*

For any $q_1, q_2, q_3 \geq 1$ there exists a graceful labeling f of $T(q_1, q_2, q_3; s)$ with $f(x) = 0$

Kotzig [14] showed that we are able to guarantee gracefulness by next two modifications of any tree.

Theorem 1.10.
- *If a leaf of a long-enough path is joined with any leaf of an arbitrary tree, the resulting tree is graceful.*
- *If a long-enough path replaces an arbitrary edge in an arbitrary tree, the resulting tree is graceful.*

Theorem 1.11. *Every tree of diameter 5 is graceful*

Sketch of a proof of this theorem could be found in survey work from Edwards and Howard [8]. Original proof is in Hrniar and Haviar 2001 [11].

Definition 1.11. *A* bamboo tree *is a rooted tree consisting of branches of equal length, the leaves of which are identified with the leaves of stars of equal size.*

Theorem 1.12. *All bamboo trees are graceful.*

Proof could be found in Sekar 2002 [21].

Definition 1.12. *An* olive tree T_k *is a rooted tree consisting of* k *branches, the* i*-th branch is a path with a length* i*. In other words –* T_k *is a spider* $S(1, 2, 3, \ldots, k)$.

Theorem 1.13. *All olive trees are graceful.*

Proof. (sketch, due to Abhyankar, Bhat-Nayak [1]) The tree T_k is labeled according to parity of k by labeling branch vertex first, with $q = (n+1)(2n+1)$ for tree T_{2n+1} or with label $q = n$ for T_{2n}. In second step are labeled the vertices adjacent to the branch vertex according to parity. In final step are labeled the remaining vertices such that sum of any two adjacent vertices is either $q - 1$ or q for first case, and q or $q + 1$ for second case. □

Definition 1.13. *The* symmetrical tree T *is a rooted tree in which every level contains vertices of the same degree.*

Theorem 1.14. *Every symmetrical tree is graceful.*

This theorem was proven by Stanton and Zarnke [20].

	α-val	β-val	σ-val	ρ-val
Caterpillars	YES	YES	YES	YES
Olive		YES	YES	YES
Symmetrical	NO	YES	YES	YES
Lobsters	NO		**YES**	YES
Spiders	NO		**YES**	**YES**
3-Cayley tree	NO			
number of leaves ≤ 4	NO	YES	YES	YES
$\lvert V(T)\rvert \leq k$	6	33	**38**	**38**
$\text{diam}(T) \leq k$	3	5	**7**	**7**

Table 1.1: State of knowledge about valuations of some well-known tree classes. "No" means that there exist at least one tree for which it is impossible to find such valuation, in most cases could be as counterexample for the α-valuations be used the spider $S(2, 2, 2)$.

There is one more classical result, currently outperformed by a distributed computing project – Graceful Tree Verification Project, which extended this result to all trees up to 33 vertices.

Theorem 1.15. *(Aldred, McKay '98) All trees with number of vertices less or equal 27 are graceful.*

This was proved by Aldred and McKay [2] in 1998. They used an algorithm, which begins with random permutation of labels $\{0, 1, \ldots, |E|\}$ and switches such pairs of vertex labels which increased cardinality of the edge labels set. If there wasn't any such pair then they used other initial permutation. They used some other heuristics to obtain the graceful labeling very fast. Time consumption for one tree needed to be very low, because the number of various trees is growing very fast – for example there are $751,065,460$ trees of order 27.

For more information about graceful and graceful-like labelings see work from Frank van Bussel [23], survey made by Michelle Edwards and Lea Howard [8]. Reader interested in labeling in general could find a lot of interesting informations in Dynamic Survey of Graph Labelings from Joseph A. Gallian [9] – really comprehensive survey with 180 pages and almost 800 cited papers.

Chapter 2

σ-Valuation of Spiders

Theorem 2.1. *Every spider $S(n_1, n_2, \ldots, n_k)$, with $n_i \geq 3$ for each i, has σ-Valuation.*

Notation 2.1. *We will for a spider $S(n_1, n_2, \ldots, n_k)$ use following notation: The root of this spider will be called r. For i-th path with a length n_i we will create a path G_i by deleting the vertex r. This path has length $|E_i| = m_i$ and $|V| = n_i$ vertices. We will call the vertex, which was in the original graph connected with the root, v_i. We will call A_i the bipartition which includes v_i and the other bipartition B_i. We will labels of vertices in G_i sometimes for brevity call the subblock G_i. Notation for the spider, which we want to label, will be as usual – the whole graph will be called G, $|E(G)| = n$ and $|V(G)| = m$. The constructed labeling will be for brevity called l even for edges and vertices, it will always be either mentioned, or clear from context, which one is meant.*

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SpidersLabeling(T, r)
label G_1 ... G_k with strongly graceful labeling, such that v_i = 0
for i := 1 ... k
    do scale the labeling of G_i such that l(e) = l(e) + sum_{j=0}^{j<i} m_j
move the labels of G_i "into" G_{i+2} for each i < k - 2
move the resulting blocks such that l(v_k) = 0, l(v_{k-1}) = |E| + 1 and l(r) = |E|
use (k - 1)-times the shift and insert operations for s_1 ... s_{k-1}
insert the edge (r, v_k)
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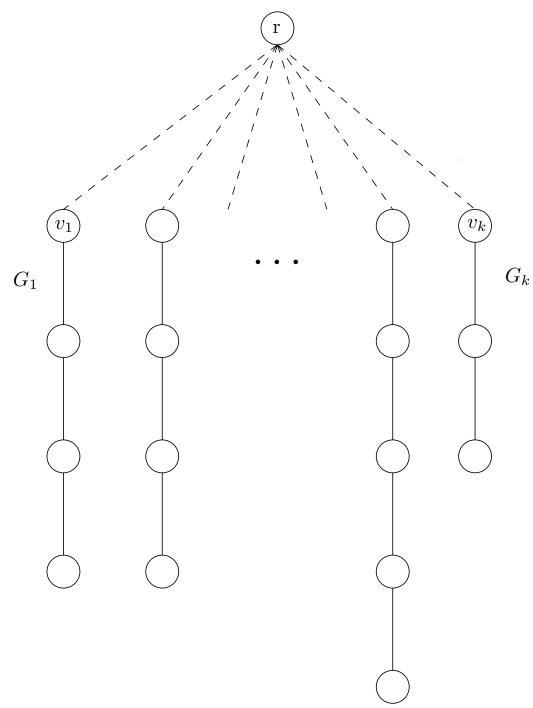

Figure 2.1: The spider $S(4, 4, \ldots, 5, 3)$ and resulting graphs $G_1 \ldots G_k$

Proof. Informal overview of the actions needed to label a spider can be seen as pseudocode in procedure SPIDERSLABELING. We will now show how could this operations be executed and that after each step some invariants hold.

Step 1: We can label each path with method shown in proof of theorem 1.2. We first label v_i with 0 and than use this alternative labeling method for achieving strongly graceful labeling.

This labeling has a positive property that we could for every i move part of vertex labels, such that we get edge labels $1, 2, \ldots, i-1, i+1, i+2 \ldots, m, m+1$, from $1, 2, \ldots, m$. This is possible to do any number of times, for any i in interval $[1, m']$, where m' is the maximal edge label used before the operation. We will use this property in following steps. We will moreover be interested just in such shifts that don't move any label of vertex in the first bipartition. It is needed to say that this property is stronger, as shift operation for bipartite labelings, where we could from k-graceful labeling, achieve an

$(k+j)$-graceful labeling for any nonnegative j.

Step 2: The possibility of scaling is a consequence of the properties discussed in last paragraph. After the scaling procedure, an invariant holds that there are used all edge labels from interval $[1, m-k]$, each of them exactly once.

Step 3: In this step we will create two "blocks", each of them containing just labels of G_i's with even/odd i. Formally we change labels for each vertex v in G_i, such that

$$l(v) = l(v) + \sum_{\substack{j \in [i+2,k] \\ j \equiv i (\mathrm{mod}\ 2)}} \lceil n_j/2 \rceil$$

This is possible without risk that we will label two vertices in one block with the same label. The reason is that we assume that each $n_i \geq 3$, what means that $m_i \geq 2$. Then the difference between highest label in i and smallest in $i+2$ is at least 3. That is enough to put labels of G_i "into" those of G_{i+2}. Besides, it is also possible to shift content of G_i inside the G_{i+2} at least once. We will use this property in Step 5. After this step we get two blocks, one of them occupying $|E|+1-k$, second one $|E|+1-k-m_k$ consecutive vertex labels (not each of this labels needs to be used).

Step 4: We "move" all labels of vertices in block containing G_{k-1} such that $l(v_{k-1}) = |E|+1$, it could be done by just adding $|E|+1$ to each label of vertex in this block. Besides, we label the vertex r with label $|E|$.

Step 5: In this step we add labels to all edges (r, v_i) except (r, v_k). We first add labels (r, v_i) for positive $i := k-1, k-3, k-5 \ldots$ then for $i := k-2, k-4, k-6 \ldots$, one label in each step. Adding an label will be done in two substeps – shift and insert. In case it will be needed, we will "move" the whole subblocks within possible bounds. Strongly graceful labeling and even our combination of such labelings has a property that an edge with lower label values have its vertex labels within interval of vertex labels of an edge with higher edge label, if both edges occupy the same block. From this property follows already mentioned property that we could always change edge labels from $l_1, l_2, \ldots l_p \wedge l_i < l_{i+1}$, such that for any $i \in [1, l_p]$ we get $l_1, l_2, \ldots, l_{r-1}, l_r + 1, l_{r+1} + 2, \ldots l_p + 1$, where l_r is the first label with value greater or equal to i. This could be done just by adding one to every vertex label higher than some s. We will call this operation *shift*. Moreover, we will call such operation *admissible* if we don't shift any vertex label in first bipartitions A_i. This is important, because than we will be sure that shift operation don't alter this bipartitions and specifically the already added edges (r, v_i).

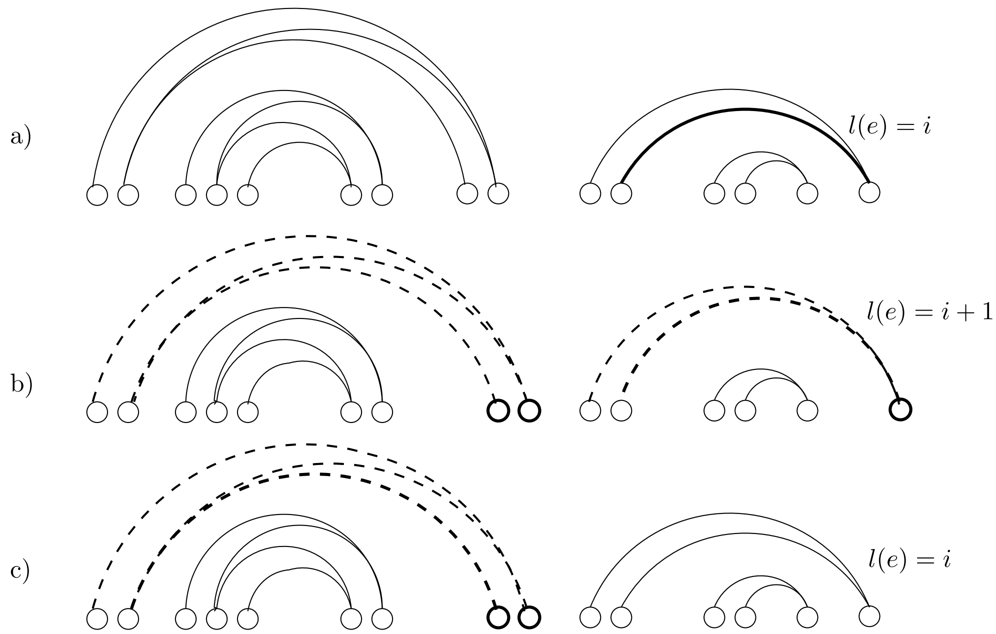

Figure 2.2: Example of the shift operation: 2.1 a) The labels of a graph G before the shift i operation, b) The graph after using the admissible shift operation for the value i, highlighted vertices are those, whose value was raised, c) Next possible admissible operation for value $i + 2$. In all cases is highlighted such edge, whose value is minimal from values greater or equal to the value of shift operation.

Now we will show a property that for any of the two blocks, there is always possible to use admissible shift operation for i or $i + 1$. We had to mention that this property isn't valid for every strongly graceful labeling – as a counterexample could be used any star, with leafs in bipartition A. But in this case are all G_i paths, what is enough for this property to hold. In general, any caterpillar, with maximum degree 2 of all vertices in bipartition B has this condition. We will generalize this result for such trees at the end of this chapter.

We will first show that this property holds for the situation before the first shift operation. In this situation there is exactly one block containing edge e with edge label $l(e) = i$. For the block without such label, for each bipartition B_j holds that all edge labels adjacent to vertices in this bipartition are either smaller or all of them are higher than i. So we could just move every vertex in all bipartitions containing just higher labels. For the bipartition with $e = (u, v) \wedge v \in B_j$ for some j could happen that we couldn't move v and all vertices with higher labels, because there is other vertex $e' = (u', v)$ with smaller label than i. But than we could use shift operation for $i+1$, because we could either move all vertices higher than v if edge with label $i + 1$ is in the same block, or it could happen that $i + 1$ is smallest label in some subblock of the other block – to move the smallest label in some subblock is always possible. Technically we do in both cases the same operation – move all vertices with higher vertex label than v in first bipartition and all the B_j bipartitions of blocks with higher or equal edge labels than $i + 1$. It could also happen that $i + 1$ is bigger than maximal used label, in this case we could just omit the shift operation, but this case wouldn't happen in our algorithm.

After any number of admissible shift operations, the situation wouldn't be much different. There could possibly happen that there exist no e with $l(e) = i$ in either of blocks, but it is not a problem, because admissible shift operation won't change the property that for any edges $e = (u, v) \wedge e' = (u', v) \wedge v \in B_j$ for any j holds $l(e) = l(e') + 1 \vee l(e) = l(e') - 1$, so there couldn't happen that e and e' with $e < i < e'$ share the same $v \in B_j$. Moving subblocks in bounds of other subblocks wouldn't affect any of this properties, because, this don't affect edge labels.

Now, when we have proven some properties for the shift operation, we could approach the main labeling scheme. We will call the subblock G_j *settled*, if we have already added label to (r, v_j), the subblock G_k will be settled by definition. We wouldn't move this blocks in any way, with exception of the shift operations done on vertices of its bipartition B_j. Other blocks would be called *unsettled*. We will add (r, v_i) labels in already mentioned order $i := k-1, k-3, k-5 \ldots, k-2, k-4, k-6 \ldots$. This order means that we will first label block containing G_{k-1}, which we will call the *right block* then the *left block*. In a case that for some $l(r, v_i)$ would be impossible to use an admissible shift operation, we will first move all unsettled subblocks in current block by 1 then use shift operation for the new value of $l(r, v_i)$ and insert this edge. As already mentioned, it is always possible to move a block within an other block by one, thanks to requirement of length of paths at least 3.

We will in each step i call procedure ShiftInsert(i,s_i :=abs($l(v_i) - l(r)$)). We first test if we could shift/insert and if not then we move the unsettled blocks. We have to treat the shift operations in right/left block differently. The reason is that the following shift operations in right block wouldn't affect the missing label s_i, because every following shift in step $j > i$ will affect just labels higher than s_j and $s_j > s_i$. In left block is the situation different – for every following shift will hold $j > i \Rightarrow s_j < s_i$ and so we have to make a "hole" in the edge labeling on such place that after all $k-1$ operations will hold that this "hole" has the same value as s_j.

We have to verify that the following holds:

1 for any i from right block and j from left holds $s_i < s_j$

2 Each shift operation lengthens the size of each block by 1

3 The move operation wouldn't cause lengthening size of any block

4 After $k-1$ operations are the "holes" in labeling, made by shift operations, matching inserted labels

To prove the first condition, we have to show that this inequality holds for the subblocks with highest(lowest) value of $s_i(s_j)$ in right(left) block. These blocks are G_1 and G_2. If we rewrite the first inequality for these two blocks, we get condition $(l(v_1) + l(v_2) < 2 \cdot |E|)$. If we look at the difference between value of $l(v_i)$ in this part of algorithm and after 3-th step, we see that only difference is, adding to one of these values $|E| + 1$ and possible $k-2$ move operations on one of both sides. After step 3 is the value of $l(v_1)+l(v_2) \leq |E|-k-2$ where k stands for at least k vertices in bipartitions B_i and 2 for additional 2 vertices that from G_1 and G_2 which has higher label (v_i has minimal label of G_i and G_i has at least 3 vertices, where we counted until now just one). So if we add to this inequality operations done until now, we get:

$$l(v_1) + l(v_2) \leq 2 \cdot |E| - 3 < 2 \cdot |E|$$

So the inequality holds and therefore also the first condition.

Second condition is evident from definition of shift operation. Third follows from an observation that we never use this operation for outer subblocks

G_k or G_{k-1} – in first case, because we don't use shift operation for G_k in this step, shift operation for G_{k-1} is used as first one, with $s_i = 1$ which allows admissible shift. Last condition was already mentioned.

So if we sum everything up, after $k-1$ operations has the highest edge label value $|E|-1$, we have assigned $|E|-1$ labels and there is no conflict between edge labels – shifting doesn't create such conflict between the labels of block edges, there is no such conflict between the labels of inserted edges, because monotonicity of s_i's in one block and the condition 1. Finally there is no conflict between the labels of block edges and inserted edges, because of condition 4. This means that after the step 6 we have used each of the edge labels $1 \dots |E|$ exactly once. Last thing that we have to show is that there was no vertex label used more than once and that no vertex label is higher than $2|E|$. There is no such conflict within blocks, and there isn't any conflict between them or with root, because after step 3 was valid an invariant that the first block has size $|E|+1-k$, what means that the highest label has value $|E|-k$, after $k-1$ shift/insert operations it has value $|E|-1$, where root has the value $|E|$. To see that, we have used just vertices $0 \dots 2|E|$ we also use invariants from step 3. There is no free space between blocks and root and they occupy $|E|$, 1 and $|E|-m_k$ vertices respectively, which is together $2|E|+1-m_k$, what means values $0 \dots 2|E|-m_k$, where $m_k \geq 2$. So the valuation we get is σ-valuation. □

SHIFTINSERT(i, s_i)

```
if shift s_i isn't admissible
      do move all unsettled subblocks in current block one step right
         if RightBlock
            then s_i := s_i + 1
            else s_i := s_i - 1
if RightBlock
   then shift s_i
   else shift s_i - (i - ⌈k/2⌉)
l(r, v_i) := s_i
```

It should be mentioned that last theorem holds not only for spiders, which are technically defined as trees having at least 3 paths meeting in vertex r. But it holds also for paths $S(n_1, n_2)$ or $S(n_1)$, or for graph with just one vertex – in this case size of both blocks will be 0. So we will treat such special cases the same way as spiders.

Theorem 2.2. *Every spider has σ-Valuation.*

Proof. We have to show that we could add to our labeling scheme any number of "legs" with lengths 1 or 2. Adding legs with length one is trivial, we just have to shift all vertex labels by one and then label the vertex of a leg with zero. We could repeat this process for each leg with length one. For adding both – the paths of length one and two we have to do 3 things – first make place for the paths of length two, with some shift and move operations. Then we could add the paths of length one and finally those of length two.

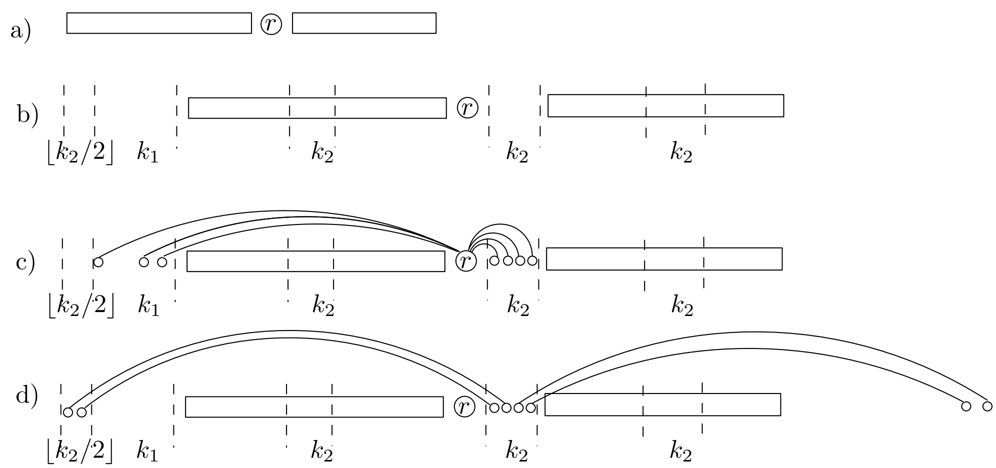

Figure 2.3: Scheme of construction used in proof of theorem 2.2 a) Labeled graph G, b) Vertex labels after shift and move operations, c) vertex labels of added vertices for paths of length 1 (left) and first labels u_i of paths of length 2 (right) and respective edges, d) vertex labels of added vertices w_i and edges (u_i, w_i)

Formally we have an already labeled graph G with k paths and m edges. We treat an graph G' with additional k_1 paths of length 1 and k_2 of length 2. After first step, we want to get a labeling with edge labels $k_2 + 1, \ldots, k_2 + m$ and with k_2 unused labels $l(r) + 1, \ldots, l(r) + k_2$. This could be done by first using k_2 shift operations with value 1, $2 \cdot k_2 + 1$ operations of move for the whole right block and then relabel r, such that $l(r) = m + k_2$. Shift operations will result into adding k_2 to all block labels. Move operations will solve vertex label conflicts and adding to (r, v_i) edge labels k_2.

Now we could add vertex labels for paths of length 1, as mentioned before – we just add one to each vertex label and label new vertex with 0. After k_1

such operations we have used edge labels $k_2+1,\dots,k_1+k_2+m$. Label of r has value $m+k_1+k_2$.

Now we will show, how we could label all paths of length 2. We will use such notation that vertex in i-th path, connected to r will be called u_i, and the leaf vertex w_i. We first have to add $\lfloor k_2/2 \rfloor$ to each vertex label, this will result to $l(r) := m+k_1+k_2+\lfloor k_2/2 \rfloor$, smallest used vertex label will be $\lfloor k_2/2 \rfloor$ and the highest will be $3\cdot k_2+\lfloor k_2/2 \rfloor+2\cdot m-m_k$ (resulting label from last theorem plus results of operations in this theorem). Then we label

$$l(u_i) := l(r)+i$$

for $1 \le i \le k_2$. Now we have used edge labels $1,\dots,k_1+k_2+m$ and we have to use labels $m+k_1+k_2+1,\dots,m+k_1+2\cdot k_2$.

Finally, we use labels for w_i as follows:

$$l(w_i) = \begin{cases} \lfloor k_2/2 \rfloor - i & \text{if } i \le \lfloor k_2/2 \rfloor, \\ 2\cdot m+2\cdot k_1+4\cdot k_2-(i-\lfloor k_2/2 \rfloor-1) & \text{if } i > \lfloor k_2/2 \rfloor, \end{cases}$$

Labelings of w_i's will use intervals $[0, \lfloor k_2/2 \rfloor - 1]$ and $[2\cdot m+2\cdot k_1+\cdot k_2+\lfloor k_2/2 \rfloor+1, 2\cdot m+2\cdot k_1+4\cdot k_2]$, so there will be no conflict with already assigned vertex labelings. We will moreover show that the resulting edge labeling will have values $[m+k_1+k_2+1, m+k_1+2\cdot k_2]$. If we calculate values of $l(u_i)-l(w_i)$ for $i \le \lfloor k_2/2 \rfloor$, we get

$$l(u_i,w_i) = m+k_1+k_2+2\cdot i$$

with minimum in $m+k_1+k_2+2$ for $i=1$ and maximum in $m+k_1+2\cdot k_2-[k_2 \text{is odd}]$ for $i=\lfloor k_2/2 \rfloor$. Similarly we will get

$$l(u_i,w_i) = m+k_1+3\cdot k_2+1-2\cdot i$$

with minimum in $m+k_1+k_2+1$ and maximum in $m+k_1+2\cdot k_2-[k_2 \text{is even}]$ for $i > \lfloor k_2/2 \rfloor$. So the resulting edge labelings are from desired interval and after this step we have successfully labeled G' with values $1,\dots,|E'|$. □

We could extend this result, without changing the construction on spider-like graphs, where we don't put together paths, but trees from a subclass of caterpillars.

Definition 2.1. *We will call* half-caterpillar *a caterpillar with start vertex of its longest path v_i and an additional property that every vertex with even distance from v_i has degree at most* 2*.*

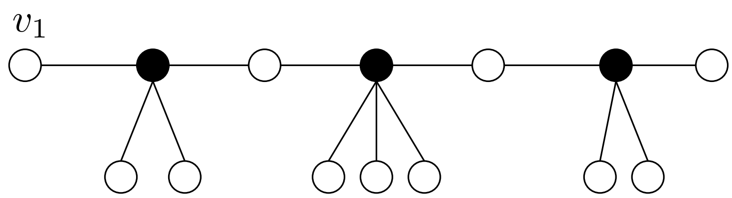

Figure 2.4: Half-caterpillar tree

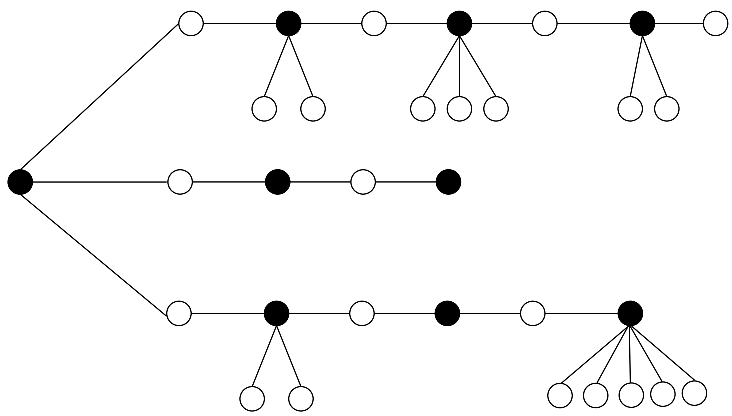

Figure 2.5: Resulting tree after joining 3 various half-caterpillars

We could see example of an half-caterpillar on figure 2.4 and example of an tree from following theorem on figure 2.5.

Theorem 2.3. *A tree obtained by merging the start vertices r of k half-caterpillars has σ-valuation.*

Proof. The proof for this class will look exactly as the proof just showed for spiders, with one exception that we don't speak about spiders with paths of length at least 3, but with half-caterpillars with at least 3 non-r-vertices. Added special cases are again just P_1 and P_2, because every other half-caterpillar has at least 3 vertices, so even second part of proof wouldn't need adding of any special cases. □

Chapter 3

σ-valuations of some other tree classes

Theorem 3.1. *Let G be a connected graph admitting graceful labeling. Then graph G' obtained by identifying each vertex v_i with the center vertex of star S_{a_i} of any size (possibly 0) has σ-valuation.*

Proof. We will take any one of the graceful labelings of G. For this holds that used vertex labels are $0, \ldots, n-1$ and that each of the edge labels $1, \ldots, n-1$ is used at least once. We will for k new edges use the labels $n, \ldots, n+k-1$. We first sort the vertices, which have to be identified with non-zero-sized stars, in ascendant order of their label values. We will then use edge labels for added edges also in ascendant order, starting with value n. So if the smallest vertex label of has value a and we will identify it with the center vertex of S_2 then we label new leafs of this star $a+n$ and $a+n+1$. It is clear that such labeling will never use same vertex label twice, because for two following nodes starting in (possibly equal) vertices with vertex labels $a_1 \leq a_2$ holds that $a_1+n+i < a_2+n+i+1$. The minimal vertex label value that we could use, is n, maximal possible vertex label is not smaller than $2(n+k-1)$ (because of connectivity we had at least $n-1$ edges in G). We wouldn't use vertex outside this boundaries, because in situation that the smallest vertex label has value 0, we get value $n := 0+n$ as new vertex label. If the highest has value $n-1$ then we get $n-1+n+k-1 \leq 2(n+k-1)$. So the resulting labeling is σ-valuation. □

Theorem 3.2. *Every lobster, irregular banana, irregular bamboo and every tree with diameter at most 7 has σ-valuation.*

Therm irregular is here used to denote possibly various star sizes in opposition of regular versions, where each star has same size.

Proof. Each of this classes could be obtained by using previous theorem on classes known to be graceful. Those classes are caterpillars for lobsters, stars $S(i, i, \ldots, i)$ for banana and bamboo trees and trees with diameter at most 5 for those with diameter 7. □

Theorem 3.3. *Every tree T with at most 38 vertices has σ-valuation.*

Proof. We will use theorem 3.1 and the properties that every tree with diameter at most 4 and with at most 33 vertices are graceful. We will for tree T with more than 33 vertices distinguish two cases. If T has at most 4 leafs, than it has graceful labeling what is also a σ-valuation. If T has more than 5 leafs, than after removing this leafs, we get tree T' with at most 33 vertices. We use theorem 3.1 for labeling the vertices from $T \setminus T'$ □

Note 3.1. *There are more possibilities how to construct σ-valuations from some more restrictive valuations. We have also proved that we could get σ-valuation from connecting any vertex v_1 of graceful graph G_1 with any vertex v_2 of G_2 admitting α-valuation. Similar result seems to be possible to obtain for graceful graph G_1 and graph G_2 with so called local bipartite labeling, if we add some restrictions on G_1 and G_2. It could be interesting to do some research in this way, mainly trying to connect more graphs with α-valuation to some graceful graph, or trying to find some construction for identifying vertices of such graphs instead of connecting them with a vertex.*

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